Mat321 – Spring 2015 –Exam3

Instructor: Dr. Francesco Strazzullo **Name** KEY

I certify that I did not receive third party help in *completing* this test (sign) ______________

Instructions. Technology and instructor's notes (including the formula sheets from our book) are allowed on this exam. Each problem is worth 10 points. Solve 10 of the 12 exercises included, as indicated: please, clearly **cross-out the two exercise you don't want to be graded.** If you use notes or formula sheets, make a reference to them. When using technology describe which one you used or print out your worksheet.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

Complete 2 of the exercises 1-3.

1. Assume that the rate of growth of a population of fruit flies is proportional to the size of the population at each instant of time. **(a)** Write this model as an ODE. **(b)** If 100 fruit flies are present initially and 200 are present after 5 days, how many will be present after 10 days?

(a) $\frac{dP}{dt} = kP \Rightarrow P = Ce^{kt}$

(b) $C = P(0) = 100 \Rightarrow P = 100e^{kt} \Rightarrow 200 = 100e^{k\cdot 5} \Rightarrow e^{5k} = 2$

$\Rightarrow 5k = \ln 2 \Rightarrow k = \frac{1}{5}\ln 2$

$P = 100e^{\frac{1}{5}\ln 2\, t} \Rightarrow P(10) = 100e^{\frac{1}{5}\ln(2)\cdot 10} = 100e^{\ln 4} = 100\cdot 4 = 400$

NOTE THAT P DOUBLES EVERY 5 DAYS.

2. Suppose a population growth is modeled by the logistic equation $\frac{dP}{dt} = 0.01P - 0.0002P^2$. Solve this differential equation with the initial condition $P(0) = 20$.

IF $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ THE GENERAL SOLUTION IS $P = \frac{M}{1 + Ae^{-kt}}$

HERE $P' = 0.01P(1 - .02P) = .01P\left(1 - \frac{P}{50}\right)$ THEN $k = .01$, $M = 50$

$P = \frac{50}{1 + Ae^{-.01t}}$ I.V. $20 = P(0) = \frac{50}{1 + A} \Rightarrow A = \frac{50}{20} - 1 = 1.5$

$P = \frac{50}{1 + 1.5e^{-0.01t}}$

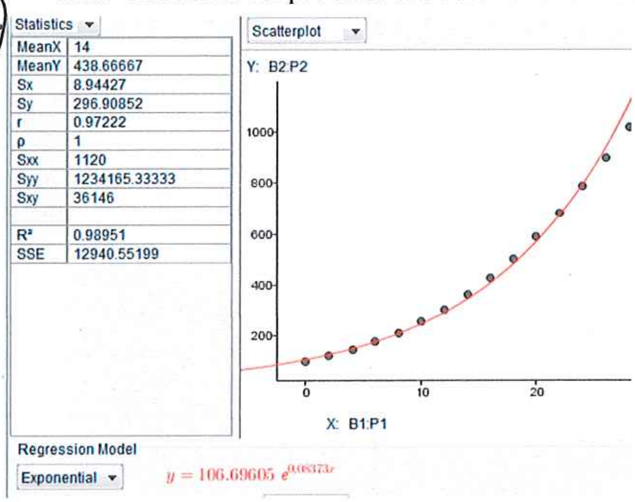
3. An outbreak of a previously unknown influenza occurred on the campus of the University of Northern South Dakota at Roscoe during the first semester. Due to the contagious nature of the disease, the campus was quarantined and the disease was allowed to run its course. The table below shows the total number P of infected students for the first four weeks of the outbreak on this campus of 2,500 students. Use technology to complete the following tasks.

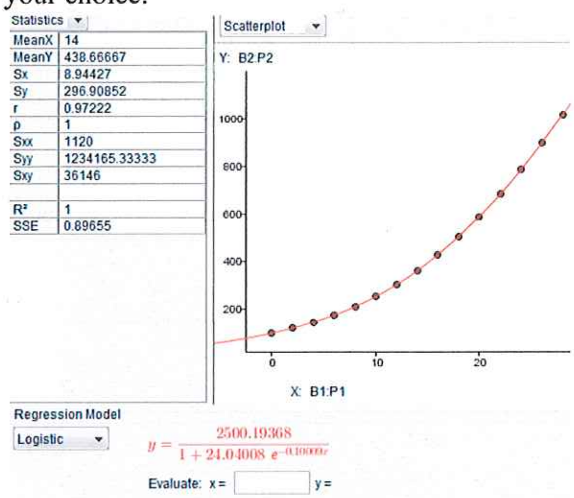
t (days)	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28
P	100	121	146	176	212	254	304	361	428	503	589	683	787	899	1017
Logistic	100	121	146	176	212	254	304	361	428	503	589	683	787	899	1017
Exponential	107	126	149	176	208	246	291	345	407	482	569	673	796	941	1113

(a) Find a logistic model and an exponential model for these data. Complete the table with predicted values using this model, using technology (you can printout or upload in EagleWeb).

(b) Compare your findings in parts (a) above. For what values would you consider both models to be a good fit for the data? Which model provides the best fit for the data? Justify your choice.

(a)





t (days)	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28
MeanVariance%	1.03	1.02	1.01	1	0.99	0.99	0.98	0.98	0.98	0.98	0.98	0.99	1.01	1.02	1.05
Variance%	-0.07	-0.04	-0.02	0	0.02	0.03	0.04	0.05	0.05	0.04	0.03	0.01	-0.01	-0.05	-0.09

(b) LOGISTIC MODEL IS A PERFECT FIT: $y = \dfrac{2500}{1 + 24e^{-0.1t}}$

NOTE THAT $M = 2500$ IS THE LIMITING SIZE

BETWEEN DAY 2 AND DAY 26 BOTH MODELS ARE A GOOD FIT.

BETWEEN DAY 6 AND DAY 22 THEIR AVERAGE IS NOT MORE THAN 1% OF THE ACTUAL VALUE.

Complete 1 of the exercises 4-5.

4. A phase portrait of a predator-prey system is given below in which F (along the vertical axis) represents the population of foxes (in thousands) and R the population of rabbits (in thousands).

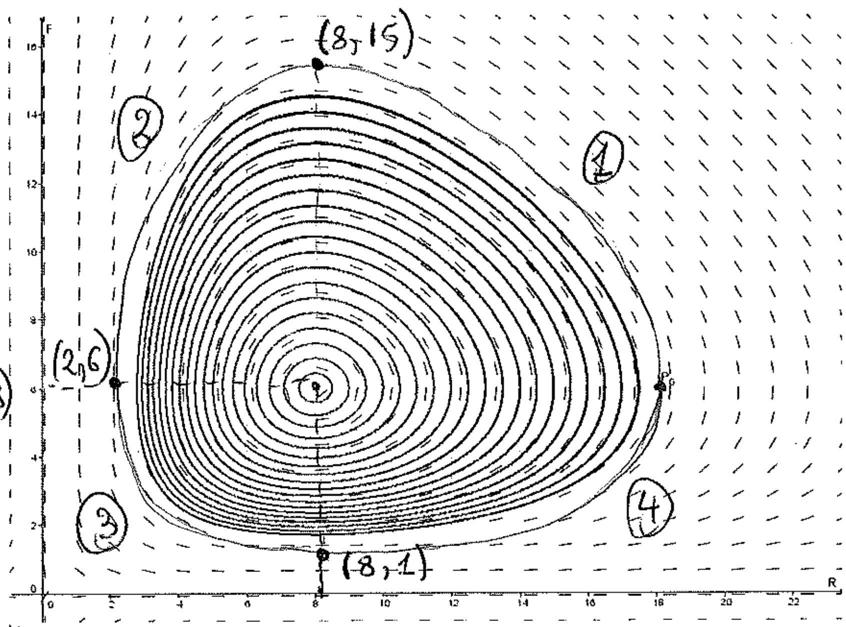
(a) Referring to the graph, what is a reasonable non-zero equilibrium solution for the system?

(b) Write down a possible system of differential equations which could have been used to produce the given graph.

(c) Describe how each population changes as time passes, using the initial condition $P_0=(18,6)$.

(d) Use your description in part (c) to make a rough sketch of the graph of R and F as functions of time.

(a) $E=(8,6)$

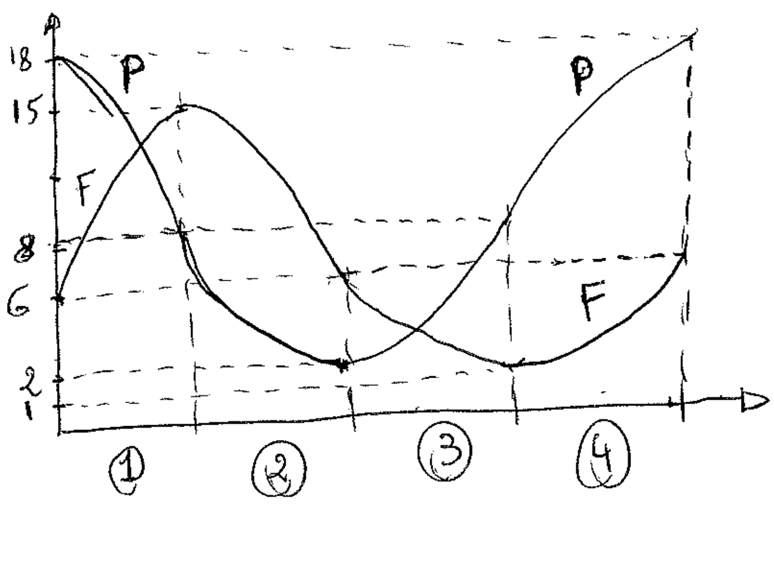


(b) $\begin{cases} R' = R(6-F) \\ F' = -F(8-R) \end{cases}$

(c) AT P_0, $F=6$ (THOUSANDS) IS AT EQUILIBRIUM AND $P=18$ IS AT MAXIMUM, THEN:

① P DECREASES TO EQUILIBRIUM $P=8$ AND F INCREASES TO MAX $F=15$; ② P KEEPS ON DECREASING TO MIN $P=2$ AND F DECREASES TO EQUILIBRIUM $F=6$; ③ P INCREASES TOWARD EQUILIBRIUM $P=8$ AND F KEEPS ON DECREASING TO MIN $F=1$; ④ F INCREASES TO EQUILIBRIUM $F=6$ AND P INCREASE TO MAX $P=18$.

(d)



5. Consider the following predator-prey system where x and y are in millions of creatures and t represents time in years:

$$\begin{cases} \dfrac{dx}{dt} = 5x - xy \\ \dfrac{dy}{dt} = -3y + xy \end{cases}$$

(a) What is the nonzero equilibrium solution?

(b) Find an expression for $\frac{dy}{dx}$.

(c) Sketch the direction field for the differential equation obtained in (b) (you can upload it in Eagleweb or print it out) and sketch a rough phase trajectory through $P_0=(10,5)$.

(d) According to your answer for part (c), on which rectangular region (that is $a<x<b$ and $c<y<d$) is $x(t)$ increasing and $y(t)$ decreasing?

(a) $x' = y' = 0$ FOR $x \neq 0$, $y \neq 0$

$0 = x' = x(5-y) \Rightarrow y = 5$

$0 = y' = -y(3-x) \Rightarrow x = 3$

$\rightarrow$ EQUILIBRIUM $(3,5)$

(b) $\dfrac{dy}{dx} = \dfrac{y'}{x'} = -\dfrac{y(3-x)}{x(5-y)}$

(c) FIGURE (GGB FILE ATTACHED)

(d) DURING PHASE 3:

$0.4 < x < 3$

$1.15 < y < .4$

Complete all the following exercises.

6. In biology, it is often assumed that the number of people that will contract a certain disease is directly proportional to the product of the number of people y who are currently infected with the disease at a particular time t and the number of people x who, at the same time, are not yet infected but who are susceptible to it. Assume that the total population, P, in the study is a constant.

(a) What does $\frac{dy}{dt}$ mean? INFECTION RATE

(b) Determine a differential equation that $y(t)$ must satisfy.

(c) What happens to y as $t \to \infty$?

(b) $P = X + Y$ = CONSTANT $\Rightarrow \begin{cases} X = P - Y \\ \frac{dX}{dt} + \frac{dY}{dt} = 0 \end{cases}$ THEN $\frac{dY}{dt} = KY \cdot X$ CAN BE WRITTEN AS $\frac{dY}{dt} = kY(P-Y)$

(c) EQUATION IN (b) IS LOGISTIC: $\frac{dY}{dt} = KP \cdot Y\left(1 - \frac{Y}{P}\right)$

THEREFORE P IS THE LIMITING SIZE OF Y: $\lim_{t \to \infty} Y(t) = P$

7. Find the orthogonal trajectories of the family of curves $2x^2 + 4y^2 = k$. Then draw several members of each family on the same coordinate plane.

SLOPE OF GIVEN FAMILY: $2 \cdot 2X + 4 \cdot 2Y \cdot Y' = 0 \Rightarrow Y' = -\frac{X}{2Y}$

SLOPE OF ORTHOGONAL TRAJECTORIES: $Y' = \frac{2Y}{X} \overset{\text{SOLVE}}{\Rightarrow} \frac{Y'}{Y} = \frac{2}{X} \Rightarrow$

$\Rightarrow \ln|Y| = 2\ln|X| + C \Rightarrow \ln|Y| = \ln X^2 + C \Rightarrow |Y| = e^{\ln X^2 + C} \Rightarrow$

$\Rightarrow |Y| = X^2 \cdot e^C \Rightarrow Y = KX^2$

8. Consider the first order ODE

$$y' = x^2 y$$

(a) Show that every member of the family of functions $y = ce^{\frac{1}{3}x^3}$ is a solution of this ODE. $Y = c\,e^{\frac{1}{3}x^3}$

(b) Find a solution of the IVP given by this ODE and the initial condition $y(0) = 8$.

(c) Find a solution of the IVP given by this ODE and the initial condition $y(8) = 0$.

(a) $y' = c\,\frac{1}{3}\cdot 3x^2\, e^{\frac{1}{3}x^3} \Rightarrow y' = c x^2 e^{\frac{1}{3}x^3} = x^2 y$ ✓

(b) $y(0) = 8 \Rightarrow 8 = c e^0 = c \Rightarrow y = 8e^{\frac{1}{3}x^3}$

(c) $y(8) = 0 \Rightarrow 0 = c\, e^{\frac{1}{3}8^3} \Rightarrow c = 0 \Rightarrow y = 0$.

9. Suppose a hail ball melts at a rate proportional to its surface area. If from $t = 0$ to $t = 6$ minutes its radius decreases from $r = 7$ cm to $r = 4$ cm, how long does it take to melt completely?

$A = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$

"BALL MELTS": $\frac{dV}{dt} = -kA$

DECREASING

$\Rightarrow -kA = \frac{dV}{dt} = \underbrace{\frac{4}{3}\pi\cdot 3r^2}_{A}\frac{dr}{dt} \Rightarrow$

$\Rightarrow \frac{dr}{dt} = -k \Rightarrow r = -kt + c$

PLUG DATA: $7 = -k\cdot 0 + c \Rightarrow c = 7$

$4 = -k\cdot 6 + c \Rightarrow -6k = 4 - 7 \Rightarrow k = \frac{3}{6} = \frac{1}{2}$ $\Rightarrow$

$\Rightarrow r = -\frac{1}{2}t + 7$

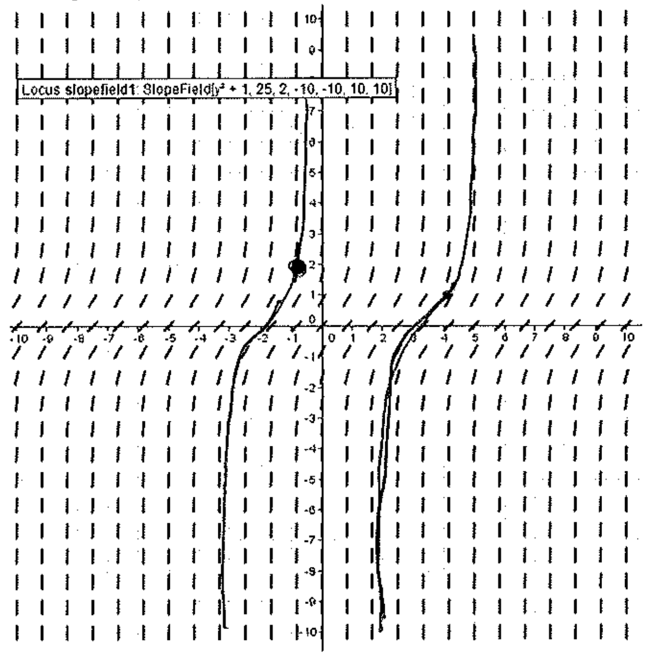
MELT COMPLETELY: $r = 0$

$\Rightarrow -\frac{1}{2}t + 7 = 0 \Rightarrow t = 14$ MINUTES

10. A direction field for a differential equation is given below:

(a) Sketch the graphs of the solutions that have initial condition (-1,2) and initial condition (4,1).

(b) Determine whether the differential equation is autonomous. Explain your answer.



(b) AUTONOMOUS BECAUSE THE SLOPE FIELD IS CONSTANT ALONG HORIZONTAL LINES

11. A population at time t is modeled by the differential equation $\frac{dP}{dt} = 0.4P\left(1 - \frac{P}{250}\right)$, where $P = P(t)$.

(a) What are the equilibrium solutions?

(b) For what values of P is the population increasing?

(c) For what values of P is the population decreasing?

(d) $P = P(t)$ is a logistic model: write the expression $P(t)$ assuming that $P(0) = 275$ individuals.

(a) EQUILIBRIUM SOLUTIONS FOR $P' = 0$: $P = 0$ OR $1 - \frac{P}{250} = 0 \Rightarrow$
$\Rightarrow P = 250$

(b) P INCREASING FOR $0 < P < 250$

(c) P DECREASING FOR $P > 250$ (POPULATION $\Rightarrow P \geq 0$)

(d) $P = \frac{M}{1 + Ae^{-kt}}$ FOR $P' = kP\left(1 - \frac{P}{M}\right)$ THEREFORE

$P = \frac{250}{1 + Ae^{-.4t}}$ PLUG $P(0) = 275 \Rightarrow 275 = \frac{250}{A+1} \Rightarrow$

$\Rightarrow A = \frac{250}{275} - 1 = -\frac{25}{275} = -\frac{1}{11} \approx -.09 \Rightarrow P \approx \frac{250}{1 - .09e^{-.4t}}$

$P = \frac{2750}{11 - e^{-.4t}}$

12. Consider the differential equation $x^2y' = y^3$.

(a) Find the general solution to the differential equation.

(b) Find the solution that satisfies the initial condition $y(1) = 1$.

(a) $x^2 \frac{dy}{dx} = y^3 \overset{y \neq 0}{\Rightarrow} \int \frac{1}{y^3} dy = \int \frac{1}{x^2} dx \Rightarrow \frac{y^{-2}}{-2} = \frac{x^{-1}}{-1} + C \Rightarrow$

$\Rightarrow \frac{1}{y^2} = \frac{2}{x} + C = \frac{2+Cx}{x} \Rightarrow y^2 = \frac{x}{2+Cx}$ (IMPLICIT GEN. SOL.)

NOTE: $y = 0 \Rightarrow y' = 0$ AND $y^3 = 0 \Rightarrow y = 0$ IS A SOLUTION.

(b) $y(1) = 1 \Rightarrow 1 = \frac{1}{2+C} \Rightarrow 2 + C = 1 \Rightarrow C = -1 \Rightarrow$

$\Rightarrow y^2 = \frac{x}{2-x}$

NOTE: $y^2 = \frac{x}{2+Cx} \geq 0 \Rightarrow \begin{cases} x \neq -2/C \\ x \geq 0 \Rightarrow \text{“}x > -\frac{2}{C} \text{ IF } C>0 \text{ OR } x < -\frac{2}{C} \text{ IF } C<0\text{”} \\ x \leq 0 \Rightarrow \text{“}x < -\frac{2}{C} \text{ IF } C>0 \text{ OR } x > -\frac{2}{C} \text{ IF } C<0\text{”} \end{cases}$

FOR THIS IVP $C = -1 < 0$ AND $-\frac{2}{C} = 2$ THEN “$x > 0 \Rightarrow$
$\Rightarrow x < 2$” AND “$x < 0 \Rightarrow x > 2$” (WHICH IS NOT POSSIBLE).
THEREFORE OUR IVP HAS DOMAIN $[0, 2)$

BECAUSE $y(1) = 1 > 0$ WE CAN FIND THE EXPLICIT SOLUTION $y = \sqrt{\frac{x}{2-x}}$ (THE OPPOSITE ONE DOESN'T SOLVE THIS IVP)