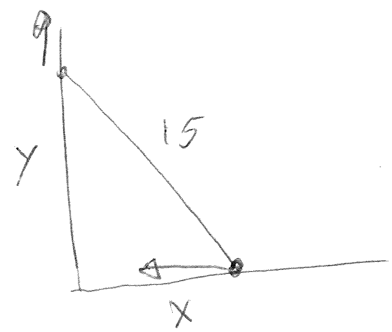
MAT 221 - Test 3 - Part 1/2 - Fall 2016

Instructor: Dr. Francesco Strazzullo My Name KEY

I certify that I did not receive third party help in *completing* this test. (*sign*)__________

Instructions. You **can not** use a graph to justify your answer. Each problem is worth 10 points. The two parts are worth 110 points.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. A ladder 15 feet long is leaning against a wall. Jake is pushing the foot of the ladder toward the wall at 2.4 feet per second. How fast is the top of the ladder sliding up the wall when the foot of the ladder is 3 feet away from the wall?



$\frac{dx}{dt} = -2.4$ FT/S (DECREASING X)

$x^2+y^2 = 15^2 \Rightarrow y = \sqrt{15^2-9} = 6\sqrt{6}$ AT GIVEN TIME.

LOOKING FOR $\frac{dy}{dt}$.

$\frac{d}{dt}\left[x^2+y^2 = 25\right] \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ PLUG DATA:

$\cancel{6}(\overset{-1.2}{-2.4}) + \overset{2}{\cancel{12}}\sqrt{6}\,\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{6}{5\sqrt{6}} = \frac{\sqrt{6}}{5} \approx .49$ FT/S

2. Find the absolute extrema and (if any) the relative extrema of the function $y = x^2 + \sin(x^2)$ on the closed interval $[0, 2]$.

$y' = 2x + 2x\cos(x^2) = 0 \Rightarrow 2x\left(1+\cos(x^2)\right) = 0 \Rightarrow$

$\Rightarrow x = 0,\quad \cos x^2 = -1 \Rightarrow x^2 = \pi + 2k\pi \Rightarrow x = \pm\sqrt{\pi + 2k\pi},$

BUT $0 \le x \le 2$ THEN $x = \sqrt{\pi} \approx 1.7725$

ABS. EXT.

x	0	$\sqrt{\pi}$	2
$f(x)$	0	3.1	3.2

$\Rightarrow$ ABS MIN AT $(0,0)$; ABS MAX AT $(2, 3.2)$

REL

$y'' = 2 + 2\cos x^2 - 4x^2\sin(x^2) \Rightarrow f''(\sqrt{\pi}) = 0$ 2ND DER. TEST CAN'T BE APPLIED.

x	1.5	1.8
$f'(x)$	$+$	$+$

($\sqrt{\pi}$ between 1.5 and 1.8) $\rightarrow$ INFLECTION AT $x = \sqrt{\pi}$

3. Let V be the volume of a right (circular) cylinder having height h and radius r, and assume that h and r vary with time. When the height is 2 in and it is increasing at a rate of 0.4 in/s, the volume is about 8 in^3 and it is decreasing at a rate of 0.15 in^3/s. $\rightarrow V' = -.15$ $\quad h' = +.4$

(a) How fast is radius of the base changing when the height is 2 in?

(b) Is the radius increasing or decreasing at that instant?

$V = \pi r^2 h \Rightarrow \dfrac{dV}{dt} = \pi\left(2r\dfrac{dr}{dt}h + r^2\dfrac{dh}{dt}\right)$

$h = 2,\ V = 8 \Rightarrow r^2 = \dfrac{4}{\pi} \Rightarrow r = 2/\sqrt{\pi}$ $\quad\Rightarrow$

$\Rightarrow -.15 = \pi\left(\dfrac{4}{\sqrt{\pi}}r' \cdot 2 + \dfrac{4}{\pi}(.4)\right) \Rightarrow$

$\Rightarrow 8\sqrt{\pi}\,r' + 1.6 = -.15 \Rightarrow r' = \dfrac{-1.25}{8\sqrt{\pi}} = \dfrac{-7}{32\sqrt{\pi}} \approx -.1234$ in/s

RADIUS DECREASING AT A RATE OF .1234 IN PER SECOND.

4. Use differentiation to find (if any) the local extrema and the inflection points of the function

$$f(x) = x - \ln\left(x^2+1\right)$$

. If there are inflection points, state if these are of highest or lowest rate.

$$f'(x) = 1 - \frac{2x}{x^2+1}$$ NEVER UNDEFINED.

$$f''(x) = -2\,\frac{(x^2+1) - x(2x)}{(x^2+1)^2} = -\frac{2(1-x^2)}{(x^2+1)^2} = \frac{2(x-1)(x+1)}{(x^2+1)^2}$$

CRIT. $\cancel{X}$: $f'(x)=0 \Rightarrow x^2+1-2x=0 \Rightarrow (x-1)^2=0 \Rightarrow x=1$

NOTE: $f'(x) = \dfrac{(x-1)^2}{x^2+1} \geq 0$ THEREFORE THERE CANNOT BE RELATIVE EXTREMA.

"INFLECTION POINTS" = "RELATIVE EXT. FOR $f'(x)$"

CANDIDATES: $x=-1$, $x=1$

x	-2	0	1.5
$f''(x)$	+	−	+
$f'(x)$	↗	↘	↗
$f(x)$	∪	∩	∪

(−1 and 1 written above the column dividers.)

THEN $x=-1$, THAT IS $(-1, f(-1)\approx -1.7)$, IS A POINT OF HIGHEST RATE, THAT IS AN INFLECTION (POINT OF DEMINISHING RETURNS).

$x=1$, THAT IS $(1, f(1)\approx .3)$, IS A POINT OF LOWEST RATE (INFLECTION POINT)

5. Find the following limits, if defined. Write the known limit or the rule for horizontal asymptotes that you use. Each exercise is worth 10 points.

(a) $\lim_{x\to 0^-} \frac{x^2}{\cos x - \cot x} =$ SUBS $\frac{0}{1-(-\infty)}$ UNDETERMINED, THEN WE REWRITE

$f(x)$

$f(x) = \frac{x^2}{\cos x - \frac{\cos x}{\sin x}} = \frac{x^2 \cdot \sin x}{\frac{1}{2}\sin(2x) - \cos x} \xrightarrow[x\to 0^-]{} \frac{0}{0-1} = 0$

$\lim_{x\to 0^-} f(x) = 0$

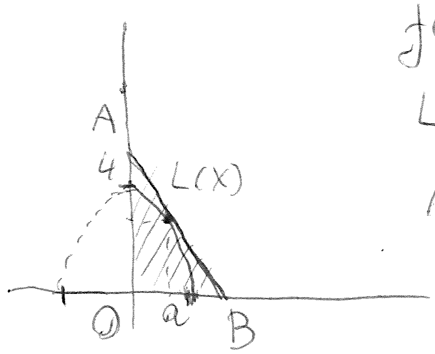
(b) $\lim_{x\to \infty} \frac{e^{-x}}{\ln x} = \frac{0}{\infty}$ UNDETERMINED, REWRITE:

$f(x) = \frac{e^{-x}}{\ln x} = \frac{1}{e^x \ln x} \xrightarrow[x\to\infty]{} \frac{1}{\infty\cdot\infty} \overset{\text{LIM}}{=} \frac{1}{\infty} \overset{\text{LIM}}{=} 0$

$\lim_{x\to\infty} f(x) = 0$

(c) $\lim_{x\to 2}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right) = \frac{1}{2-1} - \frac{1}{\ln 2} = 1 - \frac{1}{\ln 2} \approx -.4427$

6. What is the smallest possible area of the triangle that is cut off by the first quadrant and whose hypotenuse is tangent to the parabola $y = 4 - x^2$ at the same point?



$f(x) = 4 - x^2 \Rightarrow f'(x) = -2x$

$L(x) = f(a) + f'(a)(x-a) = 4 - a^2 - 2a(x-a)$

A IS THE Y-INTERCEPT OF $L(x)$ AND B IS ITS X-INTERCEPT, THEN:

$A = (0, L(0) = 4 + a^2)$

$L(x) = 0 \Rightarrow 4 + a^2 - 2ax = 0 \Rightarrow x = \frac{4+a^2}{2a} \Rightarrow B = \left(\frac{4+a^2}{2a}, 0\right)$

S = AREA OF TRIANGLE $= \frac{1}{2}\,\overline{OA}\cdot\overline{OB} = \frac{1}{2}(4+a^2)\left(\frac{4+a^2}{2a}\right) \Rightarrow$

$S = \frac{1}{4}\frac{(4+a^2)^2}{a} \Rightarrow S' = \frac{1}{4}\frac{2(2a)(4+a^2)a - (4+a^2)^2}{a^2} =$

$= \frac{1}{4a^2}(4+a^2)(4a^2 - 4 - a^2) = \frac{1}{4a}(4+a^2)(3a^2-4) \Rightarrow$

$S' = 0$ FOR $3a^2 - 4 = 0 \Rightarrow a = \pm\frac{2}{\sqrt{3}}$, $a > 0 \Rightarrow a = \frac{2}{3}\sqrt{3} \approx 1.1547$

CHECK MIN:

X	1	$2/\sqrt{3}$	2
$S'(x)$	$-$		$+$
S	↘		↗

REL. MIN

ASKED FOR $S(a) = \frac{1}{4}\frac{(4+4/3)^2}{2/\sqrt{3}} = \frac{32}{9}\sqrt{3}$

≈ 6.1584

7. Find all value(s) of c (if any) that satisfy the conclusion of the Mean Value Theorem for the function $f(x) = e^{x^2-x}$ on the interval $[-1, 2]$.

MVT (ROLLE $f(a)=f(b)$) $\Rightarrow$ $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{e^2-e^2}{3} = 0$, WITH $-1<c<2$.

$f'(x) = (2x-1)e^{x^2-x} = 0 \Rightarrow 2x-1=0$ OR $e^{x^2-x}=0 \Rightarrow$

$\Rightarrow 2x-1=0 \Rightarrow x = \frac{1}{2}$

THEN $c = \frac{1}{2}$

8. Consider the equation $e^x = 3 - 2x$. Use Newton's method to find the solution in $[0, 1]$, by completing the following steps.

(a) Write the iterative formula for x_{n+1}.

(b) Choose x_1 and compute the third approximation x_3, rounded to the fourth decimal place.

(c) Use technology to find the requested solution (preferably in symbolic form) and compare it to x_3.

WE NEED TO WRITE THE EQUATION IN STANDARD FORM: $f(x)=0$

$f(x) = e^x + 2x - 3$; $f'(x) = e^x + 2$

(a) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

(b) GRAPHICALLY, ONE CAN CHOOSE $x_1 = 1$:

$x_2 = 1 - \frac{e-1}{e+2} \approx .6358$

$x_3 \approx .5946$

(c) SYMBOLIC FORM OF SOLUTION IS NOT AVAILABLE: $x \approx .5942096$

CLOSE TO x_3 TO THE THOUSANDTHS

MAT 221 - Test 3 - Part 2/2 - Fall 2016

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Instructions. You **can not** use a graph to justify your answer. Each problem is worth 10 points, therefore your Test 3 will be graded out of 100 points.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Find (if any) the absolute and the relative extrema of the function $y = 9x^4 + 20x^3 - 12x^2$.

Y IS AN EVEN POLYNOMIAL WITH POSITIVE LEADING COEFFICIENT, THEREFORE IT HAS AN ABSOLUTE MINIMUM, CORRESPONDING TO THE SMALLEST RELATIVE MINIMUM.

$$f'(x) = 36x^3 + 60x^2 - 24x = 12x(3x^2+5x-2)$$
$$= 12x(3x-1)(x+2) = 0 \Rightarrow x = 0, \tfrac{1}{3}, -2$$

$$f''(x) = 12(9x^2 + 10x - 2)$$

$f''(0) < 0 \Rightarrow$ REL. MAX. $(0, 0)$

$f''(\tfrac{1}{3}) > 0 \Rightarrow$ REL. MIN. $(\tfrac{1}{3}, -\tfrac{13}{27})$

$f''(-2) > 0 \Rightarrow$ REL. MIN. $(-2, -64)$

ABSOLUTE MINIMUM AT $(-2, -64)$

THERE IS NONE ABS. MAX.

2. $\lim_{x\to 2^+} \left(\frac{1}{x-2} - \frac{1}{\ln(x^2-4)} \right) \overset{\text{SUB}}{\underset{\text{LIM}}{=}} \frac{1}{0^+} - \frac{1}{-\infty} \overset{\text{LIM}}{=} +\infty - 0 = +\infty$

ALTERNATIVE:

$$f(x) = \frac{1}{x-2} - \frac{1}{\ln(x^2-4)} = \frac{1}{x-2}\left(1 - \frac{(x-2)^{-1}}{\ln(x^2-4)}\right) \longrightarrow +\infty(1-0) = +\infty$$

$$(x-2)\cdot\ln(x^2-4) \xrightarrow{0\cdot\infty} \frac{\ln(x^2-4)}{\frac{1}{x-2}} \underset{HR}{\overset{\frac{\infty}{\infty}}{=}} \frac{\frac{2x}{x^2-4}}{-(x-2)^{-2}} = \frac{2x(x-2)^{\cancel{2}}}{\cancel{(x-2)}(x+2)} \longrightarrow 0$$