Mat321 – Spring 2018– TEST 4

Instructor: Dr. Francesco Strazzullo **Name** KEY

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SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Consider the recursive sequence defined by

$$x_1 = 3\,; \quad x_{n+1} = 3 - \frac{2}{x_n}\,, \qquad n > 1.$$

a) Evaluate the first three terms of this sequence.
b) You can assume the sequence to be monotonic and bounded (and hence convergent). Indicate if it is increasing or decreasing, and find a bounding interval.
c) Find the limit of this sequence.

(a) $X_1 = 3\,;\quad X_2 = 3 - \frac{2}{3} = \frac{7}{3}\,;\quad X_3 = 3 - \frac{2}{7/3} = \frac{15}{7}$

(b) $X_1 = 3\,;\ X_2 \approx 2.333\,;\ X_3 \approx 2.142$ APPEARS DECREASING

$2 \le X_n \le 3$.

(c) THE LIMIT MUST BE A SOLUTION OF $X = 3 - \frac{2}{X}$:

$X^2 = 3X - 2 \Rightarrow X^2 - 3X + 2 = 0 \Rightarrow X = \frac{3 \pm \sqrt{9-8}}{2}$

$\left.\begin{array}{l} \Rightarrow X = 2,\ 1 \\ 2 \le L \le 3 \end{array}\right] \Rightarrow L = 2$

2. Prove divergent or find the sum of the series

$$\sum_{n=1}^{\infty} \frac{2}{n(n-4)}$$

DIVERGENT, BECAUSE THE SEQUENCE OF TERMS $a_n = \frac{2}{n(n-4)}$ IS UNDEFINED FOR $n=4$.

IF WE HAD $\sum_{n=5}^{\infty} \frac{2}{n(n-4)}$ WE COULD REWRITE IT AS A TELESCOPIC SERIES USING THE PARTIAL FRACTION DECOMP. $\frac{2}{n(n-4)} = \frac{1}{2}\left(\frac{1}{n-4} - \frac{1}{n}\right)$

$$\sum_{n=5}^{\infty} \frac{2}{n(n-4)} = \frac{1}{2}\sum_{n=5}^{\infty}\left(\frac{1}{n-4} - \frac{1}{n}\right) \xrightarrow{i=n-4} = \frac{1}{2}\sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{i+4}\right)$$ THEN

$$2S_n = \sum_{i=1}^{n}\left(\frac{1}{i} - \frac{1}{i+4}\right) = 1 - \cancel{\frac{1}{5}} + \frac{1}{2} - \cancel{\frac{1}{6}} + \frac{1}{3} - \cancel{\frac{1}{7}} + \frac{1}{4} - \cancel{\frac{1}{8}} + \cancel{\frac{1}{5}} - \frac{1}{9} + \cancel{\frac{1}{6}} - \cancel{\frac{1}{10}} +$$

$$\dots + \cancel{\frac{1}{n}} - \frac{1}{n+4} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{n+4} = \frac{25}{12} - \frac{1}{n+4} \xrightarrow[n \to \infty]{} \frac{25}{12} \Rightarrow$$

$$\Rightarrow \sum_{n=5}^{\infty} \frac{2}{n(n-4)} = \lim_{n\to\infty} S_n = \frac{1}{2}\cdot\frac{25}{12} = \frac{25}{24}$$

3. Prove divergent or find the sum of the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n(n-4)}$$

SERIES IS DIVERGENT AS ABOVE, BECAUSE a_4 IS UNDEFINED.

IF WE HAD $\sum_{n=5}^{\infty} \frac{2^n}{n(n-4)}$, THEN $a_n = \frac{2^n}{n(n-4)} \xrightarrow[n\to\infty]{} \infty$

BECAUSE $\lim_{x\to\infty} \frac{2^x}{x(x-4)} = \lim_{x\to\infty} \frac{(\ln 2)\, 2^x}{2x} =$

$$= \lim_{x\to\infty} \frac{(\ln 2)^2\, 2^x}{2} = \infty$$

4. Find the radius of convergence of
$\sum_{n=1}^{\infty} \frac{(x+4)^n}{n5^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+4)^{n+1}}{(n+1)\,5^{n+1}} \cdot \frac{n5^n}{(x+4)^n}\right| =$$

$$= \frac{1}{5}|x+4| \cdot \frac{n}{n+1} \xrightarrow[n\to\infty]{} \frac{1}{5}|x+4| \underset{\text{SET}}{<} 1 \Rightarrow$$

$$\Rightarrow |x+4| < 5 \Rightarrow R = 5$$

$|x-a| < R$

5. Find the radius of convergence of
$\sum_{n=1}^{\infty} \frac{(x+4)^n}{n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+4)^{n+1}}{n+1} \cdot \frac{n}{(x+4)^n}\right| = |x+4| \cdot \frac{n}{n+1} \longrightarrow |x+4| \Rightarrow$$

$$|x+4| < 1 \Rightarrow R = 1$$

6. Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(2x+3)^n}{4n^2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2x+3)^{n+1}}{4(n+1)^2} \frac{4n^2}{(2x+3)^n}\right| =$$

$$= |2x+3| \cdot \frac{n^2}{(n+1)^2} \xrightarrow[n\to\infty]{} |2x+3| < 1 \Rightarrow$$

$$-1 < 2x+3 < 1 \Rightarrow -\frac{1}{2} < x - \left(-\frac{3}{2}\right) < \frac{1}{2} \quad \left(\text{RADIUS} = \frac{1}{2}\right) \Rightarrow -2 < x < -1$$

- $x = -2 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2}$ $\quad b_n = \frac{1}{4n^2}$ DECREASING INFINITESIMAL $\Rightarrow$ CONV.
- $x = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{4n^2}$ $\quad$ 2-SERIES $\Rightarrow$ CONV.

INTERVAL OF CONVERGENCE $[-2, -1]$

7. Find the coefficient of x^5 in the Maclaurin series for $f(x) = x^5 - \cos(3x)$.

$$f'(x) = 5x^4 + 3\sin(3x) \; ; \; f''(x) = 20x^3 + 9\cos(3x)$$

$$f^{(3)}(x) = 60x^2 - 27\sin(3x) \; ; \; f^{(4)}(x) = 120x - 81\cos(3x)$$

$$f^{(5)}(x) = 120 + 243\sin(3x)$$

$$\text{"COEFF. } x^5\text{"} = \frac{f^{(5)}(0)}{5!} = \frac{120}{5!} = 1$$

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1. Consider the recursive sequence defined by

$$x_1 = 2\,; \quad x_{n+1} = 3 + \frac{1}{2x_n}\,, \qquad n > 1.$$

a) Evaluate the first three terms of this sequence.
b) You can assume the sequence to be monotonic and bounded (and hence convergent). Indicate if it is increasing or decreasing, and find a bounding interval.
c) Find the limit of this sequence.

(a) $X_1 = 2$; $X_3 = \frac{41}{13} \approx 3.\overline{15384}$; $X_5 = \frac{818}{259} \approx 3.1583$
$X_2 = \frac{13}{4} \approx 3.25$; $X_4 = \frac{259}{82} \approx 3.1585$

(b) ACTUALLY, CONSIDERING $n > 0$, WE HAVE:

- $X_{2n} > X_{2(n+1)}$, SUBSEQ. OF EVEN DECREASING
- $X_{2n+1} < X_{2(n+1)+1}$, SUBSEQ. OF ODD INCREASING
- $2 \leq X_n \leq 3.25$

(c) $\{X_{2n}\}$ AND $\{X_{2n+1}\}$ MUST CONVERGE, WITH LIMIT SATISFYING THE RECURSIVE EQUATION: $X = 3 + \frac{1}{2X} \Rightarrow$

$\Rightarrow 2X^2 = 6X + 1 \Rightarrow 2X^2 - 6X - 1 = 0 \Rightarrow$

$\Rightarrow X = \frac{3 \pm \sqrt{9+2}}{2} = \frac{3 \pm \sqrt{11}}{2}$ $\left\langle \begin{array}{l} \frac{3-\sqrt{11}}{2} < 0 \text{ OUT OF BOUNDS} \\ \frac{3+\sqrt{11}}{2} \approx 3.158312 \checkmark \end{array} \right.$

$$\lim_{n\to\infty} X_n = \frac{3+\sqrt{11}}{2}$$

2. Prove divergent or find the sum of the series

$$\sum_{n=1}^{\infty} \frac{3}{n(n+2)} = 3\sum_{n=1}^{\infty} \frac{1/2}{n} - \frac{1/2}{n+2} = \frac{3}{2}\sum_{i=1}^{\infty} \frac{1}{i} - \frac{1}{i+2}$$

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} \Rightarrow (n+2)A + nB = 1 \Rightarrow \begin{cases} 2A = 1 \Rightarrow A = \frac{1}{2} \\ -2B = 1 \Rightarrow B = -\frac{1}{2} \end{cases}$$

$$S_n = \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+2} = 1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} + \dots + \cancel{\frac{1}{n}} - \frac{1}{n+2}$$

$$= 1 + \frac{1}{2} - \frac{1}{n+2} \xrightarrow[n\to\infty]{} \frac{3}{2} \quad \text{THEN}$$

$$\sum_{n=1}^{\infty} \frac{3}{n(n+2)} = \frac{3}{2} \cdot \frac{3}{2} = \frac{9}{4}$$

3. Find the coefficient of x^3 in the Taylor series for $f(x) = \sin(2\pi x)$ around $x = 1$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{WITH } a = 1$$

$$f^{(1)}(x) = 2\pi\cos(2\pi x);\ f^{(2)}(x) = -(2\pi)^2\sin(2\pi x);$$

$$f^{(3)}(x) = -(2\pi)^3\cos(2\pi x);\ f^{(4)}(x) = (2\pi)^4\sin(2\pi x);$$

$$f^{(1)}(1) = 2\pi;\ f^{(2)}(1) = 0;\ f^{(3)}(1) = -(2\pi)^3;\ f^{(4)}(1) = 0$$

$$T_{f,1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2\pi)^{2n-1}}{(2n-1)!}(x-1)^{2n-1} = 2\pi(x-1) - \frac{(2\pi)^3}{3!}(x-1)^3 + \dots$$

BINOMIAL THEOREM: $(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k} \Rightarrow$ COEFF. OF x^3

IN $(x-1)^m$ IS $\binom{m}{3} = \frac{m!}{(m-3)!\,3!} = \frac{m(m-1)(m-2)}{3!}$ THEN

COEFF. OF x^3 IS $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}(2\pi)^{2n-1}}{(2n-1)!} \cdot \frac{(2n-1)(2n-2)(2n-3)}{3!} =$

$$= \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(2\pi)^{2n-1}}{3!\,(2n-4)!}$$