# MAT 221 - Test 3 - Part 1/2 - Fall 2015

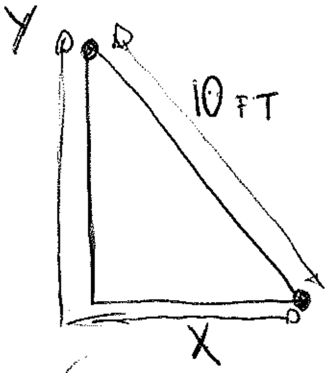
Instructor: Dr. Francesco Strazzullo

My Name KEY

I certify that I did not receive third party help in *completing* this test. (*sign*)

**Instructions.** You **can not** use a graph to justify your answer. Each problem is worth 10 points. The two parts are worth 110 points.
**SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).**

1. A ladder 10 feet long is leaning against a wall. If the foot of the ladder is being pulled away from the wall at 3 feet per second, how fast is the top of the ladder sliding down the wall when the foot of the ladder is 8 feet away from the wall?



AT GIVEN TIME $\begin{cases} x' = \dfrac{dx}{dt} = 3 \text{ FT/S} & \text{(INCREASING)} \\ x = 8 \text{ FT} \Rightarrow y = \sqrt{100-64} = 6 \end{cases}$

$x^2 + y^2 = 10^2$ ⇒ (DIFFER. W.R.T $t$)

(GEN. PWR RULE) ⇒ $2xx' + 2yy' = 0$ ⇒ (PLUG DATA)

AT GIVEN TIME: $2(8)(3) + 2(6)y' = 0$ ⇒

$$\Rightarrow y' = -\frac{6 \cdot 8}{6 \cdot 2} = -4 \text{ FT/S}$$ (DECREASING HEIGHT ✓)

2. Find the absolute extrema and (if any) the relative extrema of the function $y = x - \cos x$ on the closed interval $[0, \pi]$.

$f(x)$ CONT. ON CLOSED INTERVAL $\Rightarrow$ THERE ARE ABS EXTREMA.

$f'(x) = 1 + \sin x$ (ALWAYS DEFINED) $\underset{\#\text{'s}}{\overset{\text{CRIT.}}{\Rightarrow}}$ $1 + \sin x = 0 \Rightarrow \sin x = -1$,
BUT ON $[0, \pi]$ $\sin x \geq 0$ $\Rightarrow$ NO REL. EXTREMA $\Rightarrow$ ABS. EXTREMA ~~AT~~ ENDPOINTS. ON $[0, \pi]$ $f'(x) > 0$ THEN WE KNOW THAT:

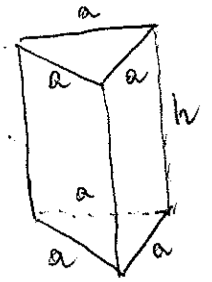
ABS MIN : $(0, f(0) = -1)$

ABS MAX : $(\pi, f(\pi) = \pi + 1 \approx 4.14)$

3. Let $V$ be the volume of a right triangular cylinder having height $h$ and base an equilateral triangle with side $a$, and assume that $h$ and $a$ vary with time. When the height is 3 in and it is decreasing at a rate of 0.6 in/s, the volume is about 7.5 in$^3$ and it is increasing at a rate of 0.25 in$^3$/s.

(a) How fast is a side of the base changing at that instant?

(b) Are the sides increasing or decreasing at that instant?



$h = 3$ , $h' = -.6$ in/s (DECR.)

$V = 7.5$ , $V' = .25$ in$^3$/s (INCR.)

LOOKING FOR $a'$ :

AREA EQUIL. TR. : $A = \frac{1}{2} a^2 \sin 60° = \frac{\sqrt{3}}{4} a^2$

$V = A \cdot h = \frac{\sqrt{3}}{4} a^2 h \Rightarrow V' = \frac{\sqrt{3}}{4}(2aa' \cdot h + a^2 h')$

PLUG DATA: $.25 = \frac{\sqrt{3}}{4}(2aa'(3) + a^2(-.6))$

$7.5 = \frac{\sqrt{3}}{4} a^2 \cdot 3 \Rightarrow a^2 = \frac{10}{\sqrt{3}} \Rightarrow a = \sqrt{\frac{10}{\sqrt{3}}} = \frac{\sqrt{10}}{\sqrt[4]{3}}$ $\Bigg] \Rightarrow$

(a) $\Rightarrow \frac{1}{\sqrt{3}} = 6 \frac{\sqrt{10}}{\sqrt[4]{3}} a' - \frac{10}{\sqrt{3}} \cdot \frac{6}{10} \Rightarrow a' = \frac{7}{6\sqrt{3}} \cdot \frac{\sqrt[4]{3}}{\sqrt{10}} \approx .0886$ in/s

(b) THE SIDE IS INCREASING AT A .0886 in/s RATE.

4. Use differentiation to find the local extrema, the inflection points, and the intervals on which the function

$$f(x) = x^3 e^{-x}$$

is increasing, decreasing, concave up or down.

$$f'(x) = 3x^2 e^{-x} + x^3(-e^{-x}) = x^2(3-x)e^{-x}$$

$$f''(x) = 3(2xe^{-x} + x^2(-e^{-x})) - f'(x)$$

$$= 6xe^{-x} - 3x^2e^{-x} - 3x^2e^{-x} + x^3e^{-x}$$

$$= x(6-6x+x^2)e^{-x}$$

$f'(x)$ AND $f''(x)$ ALWAYS DEFINED.

$$f'(x) = 0 \Rightarrow \begin{cases} x^2 = 0 \Rightarrow x = 0 \\ 3 - x = 0 \Rightarrow x = 3 \end{cases}$$

0 3

| $x$ | $-1$ | $1$ | $4$ |
|---|---|---|---|
| $f'(x)$ | + | + | − |
| $f(x)$ | ↗ | ↗ | ↘ |

NOT REL. EXT MAX.

$\boxed{\text{f(x) IS INCREASING ON } (-\infty, 3) \quad \text{f(x) IS DECREASING ON } (3, +\infty)}$

$\boxed{\text{REL. MAX } \left(3, f(3) = \frac{27}{e^3} \approx 1.34\right)}$

$$f''(x) = 0 \begin{cases} x = 0 \\ x^2 - 6x + 6 = 0 \Rightarrow x = 3 \pm \sqrt{3} \approx \begin{cases} 1.3 \\ 4.7 \end{cases} \end{cases}$$

0 $3-\sqrt{3}$ $3+\sqrt{3}$

| $x$ | $-1$ | $1$ | $2$ | $9$ |
|---|---|---|---|---|
| $f''(x)$ | − | + | − | + |
| $f'(x)$ | ↘ | ↗ | ↘ | ↗ |
| $f(x)$ | ∩ | ∪ | ∩ | ∪ |

LOWEST RATE HIGHEST RATE LOWEST RATE

$f(x)$ IS CONCAVE DOWN ON $(-\infty, 0) \cup (3-\sqrt{3}, 3+\sqrt{3})$

$f(x)$ IS CONCAVE UP ON $(0, 3-\sqrt{3}) \cup (3+\sqrt{3}, \infty)$

$(0,0)$ AND $\left(3+\sqrt{3}, \frac{(3+\sqrt{3})^3}{e^{3+\sqrt{3}}} \approx .93\right)$ INFLECTION POINTS OF LOWEST RATE

$\left(3-\sqrt{3}, \frac{(3-\sqrt{3})^3}{e^{3-\sqrt{3}}} \approx .57\right)$ INFLECTION POINT OF HIGHEST RATE.

5. Find the following limits, if defined. Write the known limit or the rule for horizontal asymptotes that you use. Each exercise is worth 10 points.

(a) $\lim\limits_{x\to 0+} \dfrac{x}{\sin x + \tan x} \underset{\substack{\uparrow \\ \text{HR.}}}{\overset{\frac{0}{0}}{=}} \lim\limits_{x \to 0^+} \dfrac{1}{\cos x + \tan^2 x + 1} = \dfrac{1}{1+0+1} = \dfrac{1}{2}$

(b) $\lim\limits_{x\to 0^+} \dfrac{e^{x^2} - 1}{x} \underset{\substack{\uparrow \\ \text{HR}}}{\overset{\frac{0}{0}}{=}} \lim\limits_{x \to 0^+} \dfrac{2xe^{x^2}}{1} = \dfrac{0}{1} = 0$

(c) $\lim_{x\to 2}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)\overset{\text{SUBS}}{\downarrow}{=} \frac{1}{2-1}-\frac{1}{\ln 2}= 1-\frac{1}{\ln 2}\approx -.4427$

6. There are 10 apple trees in an orchard. Each tree produces 50 lbs of apples. For each additional tree planted in the orchard, the output per tree drops by 2 lb.

   (a) How many trees should be added to the existing orchard in order to maximize the total output of apples?

   (b) When is the output of apples increasing *most rapidly*?

$X =$ "NUMBER OF TREES ADDED TO THE ORIGINAL 10"

$Y =$ "TREES OUTPUT" = "# OF TREES" · "OUTPUT PER EACH TREE" $\Rightarrow$

$\Rightarrow Y = (10+X)\cdot(50-2X) = 500 - 20X + 50X - 2X^2$

$Y = 500 + 30X - 2X^2$. SAY $Y = f(x)$

(a) $f(x)$ IS A DOWNWARD PARABOLA, SO THE MAX IS AT VERTEX $X = -\frac{b}{2a}$

USING CALCULUS: $f'(x) = 30 - 4X = 0 \Rightarrow X = \frac{30}{4} = \frac{15}{2} = 7.5$, THEN $f''(x) = -4 < 0 \Rightarrow$ MAXIMUM. BECAUSE WE CAN'T ADD 7.5 TREES, WE NEED TO CHECK IF MAX IS FOR $X = 7$ OR $X = 8$:

$f(7) = 612$, $f(8) = 612 \Rightarrow$ EITHER 7 OR 8 TREES.

(b) MAX FOR $f'(x)$: $f''(x) = -4$ CONSTANTLY, $f'(x)$ IS ALWAYS DECREASING THEREFORE THE MAX IS AT THE BEGINNING: $X = 0$.

7. Find all value(s) of $c$ (if any) that satisfy the conclusion of the Mean Value Theorem for the function $f(x) = \frac{1}{1+x}$ on the interval $[0, 1]$.

MVT: THERE IS $c$ IN $[a,b]$ SUCH THAT $f'(c) = \frac{f(b)-f(a)}{b-a} =$

$$= \frac{\frac{1}{2}-1}{1-0} = -\frac{1}{2} \quad \Big] \Rightarrow c \text{ IS A SOLUTION OF } \frac{1}{(1+x)^2} = \frac{1}{2} \Rightarrow$$

$$f'(x) = -\frac{1}{(1+x)^2}$$

$\Rightarrow (x+1)^2 = 2 \Rightarrow x = -1 \pm \sqrt{2}$, BUT ONLY THE POSITIVE ONE IS IN $[0,1] \Rightarrow c = -1 + \sqrt{2} \approx .4142$

8. Consider the equation $6x^3 - x^2 - 19x - 6 = 0$. Use Newton's method to find the solution in $[-1,1]$, by completing the following steps.

(a) Write the iterative formula for $x_{n+1}$.

(b) Choose $x_1$ and compute the third approximation $x_3$, rounded to the fourth decimal place.

(c) Use technology to find the requested solution (preferably in symbolic form) and compare it to $x_3$.

(a) APPLIED TO $f(x) = 0$: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

(b) $f'(x) = 18x^2 - 2x - 19$. USE $x_1 = 0$ * (ORIGINALLY IT WAS ON $[0,1]$, BECAUSE OF A TYPO, BUT THERE ARE NO SOLUTIONS THERE)

$$x_2 = 0 - \frac{f(0)}{f'(0)} \approx -.3158$$

$$x_3 = -.3158 - \frac{f(-.3158)}{f'(-.3158)} \approx -.3332$$

(c) USING A CAS (GGB): $x = -\frac{3}{2}, -\frac{1}{3}, 2$.

IN $[-1,1]$, $x = -\frac{1}{3}$.

# MAT 221 - Test 3 - Part 2/2 - Fall 2015

Instructor: Dr. Francesco Strazzullo

Name KEY

**Instructions.** You **can not** use a graph to justify your answer. This problem is worth 10 points, therefore your Test 3 will be graded out of 110 points.
**SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).**

Find (if any) the absolute and the relative extrema of the function $y = x^2e^{-x^2+3x+1}$.

$$f'(x) = 2x\,e^{-x^2+3x+1} + x^2(-2x+3)e^{-x^2+3x+1}$$

$$= x(2-2x^2+3x)\,e^{-x^2+3x+1}$$

CRIT. #'S:

$f'(x)$ ALWAYS DEFINED. $f'(x)=0 \begin{cases} x=0 \\ 2x^2-3x-2=0 \\ (2x+1)(x-2)=0 \begin{cases} x=-\frac{1}{2} \\ x=2 \end{cases} \end{cases}$

$1^{ST}$ DER. TEST

$-\frac{1}{2}$ $0$ $2$

| X | -1 | -.1 | 1 | 3 |
|---|---|---|---|---|
| $f'(x)$ | + | − | + | − |
| $f(x)$ | ↗ | ↘ | ↗ | ↘ |

MAX MIN MAX

$f(-\frac{1}{2}) = \frac{1}{4}e^{-3/4} \approx .118$

$f(0) = 0$

$f(2) = 4e^3 \approx 80.342$

NOTE THAT $f(x)$ IS ALWAYS NON-NEGATIVE: $f(x) \geq 0$, THEREFORE

$(0, 0)$ REL. AND ABS. MINIMUM

$(-\frac{1}{2}, \frac{1}{4}e^{-3/4})$ AND $(2, 4e^3)$ REL. MAXIMA.