MAT 320 – Spring 2019 – Exam3

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I certify that I did not receive third party help in *completing* this test (sign) ________________

Instructions. Technology is allowed on this exam. Each problem is worth 10 points. If you use formulas or properties from our book, make a reference. You are expected to use a CAS for some computations, then upload your files in Eagleweb.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1) Let $H = \{a + bx^2 + cx^3 \in P_3 \text{ such that } a = b + c\}$ be a subset of the real vector space of polynomials of degree at most 3. If H is a subspace of P_3 then provide one of its bases, otherwise show at least one property of subspaces that H does not satisfy.

let $p_1 = a_1 + b_1x^2 + c_1x^3 = b_1 + c_1 + b_1x^2 + c_1x^3$ AND
$p_2 = b_2 + c_2 + b_2x^2 + c_2x^3$ BE IN H, AND t_1, t_2 BE REAL.

$$t_1p_1 + t_2p_2 = \left(t_1(b_1+c_1) + t_1b_1x^2 + t_1c_1x^3\right) + \left(t_2(b_2+c_2) + t_2b_2x^2 + t_2c_2x^3\right)$$

$$= \left(t_1(b_1+c_1) + t_2(b_2+c_2)\right) + \underbrace{(t_1b_1 + t_2b_2)}_{b_3}x^2 + \underbrace{(t_1c_1+t_2c_2)}_{c_3}x^3$$

$$= \underbrace{\left((t_1b_1+t_2b_2) + (t_1c_1+t_2c_2)\right)}_{b_3+c_3} + b_3x^2 + c_3x^3 \quad \text{IN } H.$$

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THEN H IS CLOSED UNDER LINEAR COMBINATIONS.

ENOUGH TO PROVE SUBSPACE

p IN H IF AND ONLY IF $p = (b+c) + bx^2 + cx^3 =$
$= b(1+x^2) + c(1+x^3) \Leftrightarrow H = \text{SPAN}(1+x^2, 1+x^3)$

A BASIS FOR H IS $\{1+x^2, 1+x^3\}$

2) Let $\mathcal{C} = \left\{C_1 = \begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}, C_2 = \begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}, C_3 = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, C_4 = \begin{bmatrix}2 & 0\\ 3 & -1\end{bmatrix}\right\}$ be a subset of the real vector space of the 2-by-2 matrices $M_{2,2}$.

(a) Use the standard basis $\mathcal{E} = \left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$ to reduce $\mathcal{C}$ to a basis $\bar{\mathcal{C}}$ of $\langle \mathcal{C} \rangle$ (Hint: for each $C_i \in \mathcal{C}$ write C_i as a linear combination of the vectors of $\mathcal{E}$.)

(b) Use part (a) to extend $\bar{\mathcal{C}}$ to a basis for $M_{2,2}$.

LET $\mathcal{E} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$; THEN: $C_1 = 1E_{11} + 2E_{21} - E_{22}$; $C_2 = 2E_{11} + E_{12} + E_{21}$; $C_3 = E_{11} + E_{12}$; $C_4 = 2E_{11} + 3E_{21} - E_{22}$.

$\vec{0} = t_1 C_1 + t_2 C_2 + t_3 C_3 + t_4 C_4$ IF AND ONLY IF $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}$

IS A SOLUTION OF $C\vec{x} = \vec{0}$, WHERE

$$C = \begin{array}{c} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 3 \\ -1 & 0 & 0 & -1 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ C_1 & C_2 & C_3 & C_4 \end{matrix} \end{array}. \qquad RREF(C) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

FIRST THREE COLUMNS INDEPENDENT; $C_4 = C_1 + C_2 - C_3$

(a) $\bar{\mathcal{C}} = \{C_1, C_2, C_3\}$

(b) LOOKING AT RREF(C), E_{22} COMPLETES TO A BASIS:

$\mathcal{B} = \{C_1, C_2, C_3, E_{22}\}$.

3) Consider $\mathcal{C} = \{\mathbf{p}_1 = x^2,\ \mathbf{p}_2 = 3x, \mathbf{p}_3 = x + x^3, \mathbf{p}_4 = 2 + x^3\}$ in P_3, the real vector space of polynomials of degree at most 3.

(a) Use the standard basis $\mathcal{E} = \{1, x, x^2, x^3\}$ to prove that $\mathcal{C}$ is a basis for P_3 (***write down what the definition of a basis is and which theorem you use to justify your answer***).

(b) (HONOR only) Write $1 - 3x + x^2 - 2x^3$ as a linear combination of the vectors in $\mathcal{C}$.

(a) BASIS = "LINEARLY INDEPENDENT SET OF GENERATORS"

IF $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$ THEN: $\vec{p}_1 = \vec{e}_3$; $\vec{p}_2 = 3\vec{e}_2$; $\vec{p}_3 = \vec{e}_2 + \vec{e}_4$; $\vec{p}_4 = 2\vec{e}_1 + \vec{e}_4$. THEN $\vec{0} = t_1\vec{p}_1 + \dots + t_4\vec{p}_4$ IF AND ONLY IF $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}$ IS A SOLUTION OF $\vec{0} = C\vec{x}$, WHERE $C = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

$$\det(C) \overset{\text{1ST ROW}}{=} -2\begin{vmatrix} 0 & 3 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \overset{\text{3RD ROW}}{=} -2(1)\begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} = -2(1)(-3) = 6 \neq 0 \Rightarrow$$

C IS NOT SINGULAR $\Rightarrow$ $C\vec{x} = \vec{0}$ ONLY FOR $\vec{x} = \vec{0}$.

THEREFORE $\vec{t} = \vec{0}$ AND $\mathcal{C}$ IS LIN. INDEP. MAXIMAL SET (THEOREM NME 6)

(b) EQUIVALENT SO SOLVING $C\vec{x} = \begin{bmatrix} 1 \\ -3 \\ 1 \\ -2 \end{bmatrix} \Rightarrow \vec{x} = C^{-1}\begin{bmatrix} 1 \\ -3 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 & 0 & 6 & 0 \\ 1 & 2 & 0 & -2 \\ -2 & 0 & 0 & 4 \\ 2 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \\ 1 \\ -2 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ -1/6 \\ -5/3 \\ 1/3 \end{bmatrix} \Rightarrow 1 - 3x + x^2 - 2x^3 = \vec{p}_1 - \frac{1}{6}\vec{p}_2 - \frac{5}{3}\vec{p}_3 + \frac{1}{3}\vec{p}_4$

4) Consider $\mathcal{C} = \{\boldsymbol{p}_1 = 1,\ \boldsymbol{p}_2 = 3x, \boldsymbol{p}_3 = x + x^2, \boldsymbol{p}_4 = 2 + x^3\}$ in $P_3[0,1]$, the real vector space of polynomials of degree at most 3 over the closed interval $[0,1]$, with inner product $\boldsymbol{p} \cdot \boldsymbol{q} = \int_0^1 \boldsymbol{p}(x)\,\boldsymbol{q}(x)\,dx$.

Use the Gram-Schmidt procedure to orthogonalize $\mathcal{C}$. Check your results.

(HONOR ONLY) Generate an orthonormal basis. Check your results.

WRITE $\mathcal{C}' = \{\vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4\}$ THE ORTHOGONALIZATION OF $\mathcal{C}$.

$$\vec{q}_1 = \vec{p}_1 \ ; \quad \vec{q}_2 = \vec{p}_2 - \frac{\vec{q}_1 \cdot \vec{p}_2}{\vec{q}_1 \cdot \vec{q}_1}\vec{q}_1 = 3x - \frac{3}{2}\cdot(1) = 3x - \frac{3}{2}$$

$$\vec{q}_1 \cdot \vec{q}_1 = \int_0^1 1^2\,dx = 1 \ ; \quad \vec{q}_1 \cdot \vec{p}_2 = \int_0^1 3x\,dx = \frac{3}{2}[x^2]_0^1 = \frac{3}{2}$$

$$\vec{q}_3 = \vec{p}_3 - \frac{\vec{q}_1 \cdot \vec{p}_3}{\vec{q}_1 \cdot \vec{q}_1}\vec{q}_1 - \frac{\vec{q}_2 \cdot \vec{p}_3}{\vec{q}_2 \cdot \vec{q}_2}\vec{q}_2 = x + x^2 - \frac{5}{6} - \frac{4}{3}\cdot\frac{1}{2}\left(3x - \frac{3}{2}\right) = \frac{1}{6} - x + x^2$$

$$\vec{q}_1 \cdot \vec{p}_3 = \int_0^1 x + x^2\,dx = \left[x^2/2 + x^3/3\right]_0^1 = \frac{5}{6} \ ; \quad \vec{q}_2 \cdot \vec{q}_2 = \int_0^1 \left(3x - \frac{3}{2}\right)^2 dx =$$

$$= \frac{1}{3}\int_0^1 3\left(3x - \frac{3}{2}\right)^2 dx = \frac{1}{3}\left[\frac{\left(3x - \frac{3}{2}\right)^3}{3}\right]_0^1 = \frac{3}{4} \ ; \quad \vec{q}_2 \cdot \vec{p}_3 = \int_0^1 \left(3x - \frac{3}{2}\right)(x + x^2)\,dx = \frac{1}{2}$$

$$\vec{q}_4 = \vec{p}_4 - \frac{\vec{q}_1 \cdot \vec{p}_4}{\vec{q}_1 \cdot \vec{q}_1}\vec{q}_1 - \frac{\vec{q}_2 \cdot \vec{p}_4}{\vec{q}_2 \cdot \vec{q}_2}\vec{q}_2 - \frac{\vec{q}_3 \cdot \vec{p}_4}{\vec{q}_3 \cdot \vec{q}_3}\vec{q}_3 \underset{\text{TECH}}{=} -\frac{1}{20} + \frac{3}{5}x - \frac{3}{2}x^2 + x^3$$

$$\vec{q}_1 \cdot \vec{p}_4 = \frac{9}{4} \ ; \quad \vec{q}_2 \cdot \vec{p}_4 = \frac{9}{40} \ ; \quad \vec{q}_3 \cdot \vec{p}_4 = \frac{1}{120} \ ; \quad \vec{q}_3 \cdot \vec{q}_3 = \frac{1}{180}$$

$$\mathcal{C}' = \left\{1,\ -\frac{3}{2} + 3x,\ \frac{1}{6} - x + x^2,\ -\frac{61}{20} + \frac{28}{5}x - \frac{13}{2}x^2 + x^3\right\}$$

HONOR $\quad \vec{q}_4 \cdot \vec{q}_4 = \dfrac{1}{2800}$, DEFINE $\vec{u}_i = \dfrac{1}{\|\vec{q}_i\|}\vec{q}_i$, WHERE

$$\|\vec{q}\| = \sqrt{\vec{q} \cdot \vec{q}} = \sqrt{\int_0^1 [q(x)]^2\,dx}. \quad \vec{u}_1 = \vec{q}_1 = 1 \ ; \quad \vec{u}_2 = -\sqrt{3} - 2\sqrt{3}x \ ;$$

$$\vec{u}_3 = 6\sqrt{5}\cdot\vec{q}_3 \ ; \quad \vec{u}_4 = 20\sqrt{7}\,\vec{q}_4$$

5) Consider $A = \begin{bmatrix} 3 & -1 & 0 & 1 \\ 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \end{bmatrix}$. You can use a CAS only to compute determinants, factor polynomials, or row-reduce matrices, to complete the following steps (**each** worth **10 points**).

a. Compute the eigenvalues of A.

b. For each eigenvalue λ, find a basis of the corresponding eigenspace and state the geometric multiplicity of λ.

a) $\det(A-\lambda I) = (1-\lambda)\begin{vmatrix} 3-\lambda & -1 & 1 \\ 1 & 2-\lambda & -2 \\ 1 & 1 & -1-\lambda \end{vmatrix} = \lambda(\lambda-1)^2(\lambda-3) \Rightarrow \lambda = 0, 1, 3.$

b) • $V_0 = Null(A) = SPAN\left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right) \Rightarrow$ "GEOM. MULT. $\lambda=0$" $= 1$; $\mathcal{B}_0 = \left\{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}$

$RREF(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

• $RREF(A-I) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow V_1 = Null(A-I) = SPAN\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) \Rightarrow G.A. = 1$; $\mathcal{B}_1 = \left\{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

• $RREF(A-3I) = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow V_3 = Null(A-3I) = SPAN\left(\begin{bmatrix} 6 \\ 2 \\ -1 \\ 2 \end{bmatrix}\right) \Rightarrow G.A. = 1$; $\mathcal{B}_3 = \left\{\begin{bmatrix} 6 \\ 2 \\ -1 \\ 2 \end{bmatrix}\right\}$

6) Let $char(B) = \lambda^6 + \lambda^5 - \lambda^4 - \lambda^3$. Determine the eigenvalues of B and state their algebraic multiplicities. You **cannot use a CAS.**

$Char(B) = \lambda^5(\lambda+1) - \lambda^3(\lambda+1) = \lambda^3(\lambda+1)(\lambda^2-1) = \lambda^3(\lambda+1)^2(\lambda-1)$

$\lambda^3 = 0 \Rightarrow \lambda = 0$ WITH ALG. MULT. 3

$(\lambda+1)^2 = 0 \Rightarrow \lambda = -1$ WITH ALG. MULT. 2

$\lambda - 1 = 0 \Rightarrow \lambda = 1$ WITH ALG. MULT. 1

7) Let $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -2 & 0 \\ 4 & 0 & 2 \end{bmatrix}$. Check if: $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ diagonalizes A.

$PAP^{-1} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, THEN P DIAGONALIZES A (OR RATHER P^{-1})

8) You can use a CAS only to compute determinants, factor polynomials, or row-reduce matrices. Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$.

$$\det(A - \lambda I) = -(\lambda+2)^2(\lambda-4) < \begin{matrix} \lambda = -2, & \text{ALG. MULT.} = 2 \\ \lambda = 4, & \text{ALG. MULT.} = 1 \end{matrix}$$

- $RREF(A+2I) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow V_{-2} = Null(A+2I) = \text{SPAN}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$
- $RREF(A-4I) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow V_4 = Null(A-4I) = \text{SPAN}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

THEN $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ IS A(N ORTHOGONAL) BASIS OF EIGENVECTORS OF $A \Rightarrow P = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ DIAGONALIZES A

WITH $P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

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SHOW YOUR WORK NEATLY, PLEASE (**no work, no credit**).

9) Consider $A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. You can use a CAS only to compute determinants, factor polynomials, or row-reduce matrices, to complete the following steps (**each** worth **10 points**).

a. Compute the eigenvalues of A.

b. For each eigenvalue λ, find a basis of the corresponding eigenspace and state the geometric multiplicity of λ.

a) $\det(A - \lambda I_n) = \begin{vmatrix} 1-\lambda & -1 & 0 & 1 \\ 1 & 2-\lambda & 0 & 2 \\ 0 & 0 & 1-\lambda & 1 \\ 1 & 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 2-\lambda & 2 \\ 1 & 1 & 1-\lambda \end{vmatrix} \begin{matrix} 1-\lambda & -1 \\ 1 & 2-\lambda \\ 1 & 1 \end{matrix}$

$= (1-\lambda)\left((1-\lambda)^2(2-\lambda) - 2 + 1 - 2(1-\lambda) + (1-\lambda) - (2-\lambda)\right) \overset{CAS}{=}$

$= (1-\lambda)(2-\lambda)(\lambda^2 - 2\lambda - 1)$ $\to \lambda = 1$, $\lambda = 2$, $\lambda = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$

b)

- $V_1 = \mathrm{Null}(A - I_4) = \mathrm{SPAN}\left(\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1}\right) \Rightarrow g_1 =$ "GEOM. MULT. OF $\lambda = 1$" $= \dim V_1 = 1$

  $\mathrm{RREF}(A - I_4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- $V_2 = \mathrm{Null}(A - 2I_n) = \mathrm{SPAN}\left(\underbrace{\begin{bmatrix} -2 \\ 3 \\ 1 \\ 1 \end{bmatrix}}_{\vec{v}_2}\right) \Rightarrow g_2 = \dim V_2 = 1$

  $\mathrm{RREF}(A - 2I_4) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- $\mathrm{RREF}(A - (1-\sqrt{2})I_4) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \sqrt{2}-1 \\ 0 & 0 & 1 & \sqrt{2}/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow V_{1-\sqrt{2}} = \mathrm{Null}(A - (1-\sqrt{2})I_4) = \mathrm{SPAN}\left(\underbrace{\begin{bmatrix} -1 \\ 1-\sqrt{2} \\ -\sqrt{2}/2 \\ 1 \end{bmatrix}}_{\vec{v}_3}\right)$

  $g_{1-\sqrt{2}} = 1$

- $\mathrm{RREF}(A - (1+\sqrt{2})I_4) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\sqrt{2}-1 \\ 0 & 0 & 1 & -\sqrt{2}/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow V_{1+\sqrt{2}} = \mathrm{Null}(A - (1+\sqrt{2})I_4) = \mathrm{SPAN}\left(\underbrace{\begin{bmatrix} -1 \\ 1+\sqrt{2} \\ \sqrt{2}/2 \\ 1 \end{bmatrix}}_{\vec{v}_4}\right)$

  $g_{1+\sqrt{2}} = 1$

CHECK: $P = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4]$ WILL DIAGONALIZE A:

$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1-\sqrt{2} & 0 \\ 0 & 0 & 0 & 1+\sqrt{2} \end{bmatrix}$ ✓