Mat 320 – Fall 2015 –Exam1

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Instructions. Technology is allowed on this exam. Each problem is worth 10 points and all exercises must be completed. If you use formulas or properties from our book, make a reference. When using technology describe which commands (or keys typed) you used or print out and attach your worksheet.
For the in-class portion, complete four exercises following the instructions.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

Complete one of the following two exercises:

1) Consider the points $P(-2,0,3,0)$, $Q(-1, 1, 1,1)$, $R(0, -2, 0, -3)$, and $S(0, 1, 1, 1)$. Determine if P, Q, R, and S are coplanar (that is in the same plane).

COPLANAR $\Leftrightarrow$ P, Q, R, AND S ARE TIPS OF VECTORS $\vec{x} = \vec{p} + t\vec{v_1} + s\vec{v_2}$, WITH $\vec{v_1}, \vec{v_2}$ LIN. INDEP. $\Leftrightarrow$ $\text{RANK}([\overrightarrow{PQ}\ \overrightarrow{PR}\ \overrightarrow{PS}]) = 2$

$$[\overrightarrow{PQ}\ \overrightarrow{PR}\ \overrightarrow{PS}] = \begin{bmatrix} 1 & 2 & 2 \\ 1 & -2 & 1 \\ -2 & -3 & -2 \\ 1 & -3 & 1 \end{bmatrix} \overset{\text{RREF}}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{RANK IS } 3$$

(THEREFORE A HYPERPLANE). $\boxed{\text{THESE POINTS ARE NOT COPLANAR}}$

2) Find a vector equation of the plane passing through the points $P(1, -2, 3)$, $Q(0, 1, -1)$, and $R(2, -1, 0)$.

$$\vec{x} = \vec{p} + s\overrightarrow{PQ} + t\overrightarrow{PR} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + s\begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

(NOTE $\overrightarrow{PQ} \neq a\overrightarrow{PR}$, THESE VECTORS ARE LIN. INDEPENDENT)

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Complete one of the following three exercises:

3) Let $\boldsymbol{a_1} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}$, $\boldsymbol{a_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\boldsymbol{a_3} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}$. Determine whether $\boldsymbol{b}$ can be written as a linear combination of $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, and $\boldsymbol{a_3}$, if possible find such a linear combination otherwise state why it is not possible.

$t_1\vec{a}_1 + t_2\vec{a}_2 + t_3\vec{a}_3 = \vec{b}$ HAS AUGMENTED MATRIX

$$\left[\begin{array}{ccc|c} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{b} \end{array}\right] \underset{\sim}{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

PIVOT IN LAST COLUMN, THEREFORE

THE SYSTEM IS INCONSISTENT AND $\vec{b} \notin \text{Span}(\{\vec{a}_1, \vec{a}_2, \vec{a}_3\})$

4) Determine if in $\mathbb{R}^5$ $Span\left\{\begin{bmatrix}2\\3\\0\\-2\\1\end{bmatrix},\begin{bmatrix}-1\\0\\1/5\\3\\-1\end{bmatrix},\begin{bmatrix}1\\1/3\\-2\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\-1\\1/4\end{bmatrix}\right\}$ represents a line, a plane, a hyperplane, or neither of these.

$$\underbrace{\begin{bmatrix}2\\3\\0\\-2\\1\end{bmatrix}}_{\vec{v}_1},\ \underbrace{\begin{bmatrix}-1\\0\\1/5\\3\\-1\end{bmatrix}}_{\vec{v}_2},\ \underbrace{\begin{bmatrix}1\\1/3\\-2\\1\\0\end{bmatrix}}_{\vec{v}_3},\ \underbrace{\begin{bmatrix}1\\0\\0\\-1\\1/4\end{bmatrix}}_{\vec{v}_4}$$

$$RREF([\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4]) = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix} \Rightarrow RANK = 4 = 5-1,$$

THEREFORE THESE SPAN A HYPERPLANE

5) Determine homogeneous equations representing $Span\left\{\begin{bmatrix}2\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\2\\1\\-1\end{bmatrix},\begin{bmatrix}0\\5\\2\\-2\end{bmatrix},\begin{bmatrix}3\\1\\-1\\1\end{bmatrix}\right\}$. $\ni \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ ONLY IF THE FOLLOWING IS THE AUGMENTED MATRIX OF A CONSIST. SYS.

$$\left[\begin{array}{cccc|c}2&-1&0&3&x_1\\1&2&5&1&x_2\\0&1&2&-1&x_3\\0&-1&-2&1&x_4\end{array}\right] \begin{array}{l}R_2 \leftrightarrow R_1\\ R_2 \to R_1 - 2R_2\\ R_4 \to R_4 + R_3\end{array} \sim \left[\begin{array}{cccc|c}1&2&5&1&x_2\\0&-5&-10&1&x_1-2x_2\\0&1&2&-1&x_3\\0&0&0&0&x_4+x_3\end{array}\right] \begin{array}{l}R_2 \leftrightarrow R_3\\ R_3 \to R_2 + 5R_3\end{array}$$

$$\sim \left[\begin{array}{cccc|c}1&2&5&1&x_2\\0&1&2&-1&x_3\\0&0&0&-4&x_1-2x_2+5x_3\\0&0&0&0&x_4+x_3\end{array}\right]$$

ROW ECHLON FORM. THE SYSTEM IS CONSISTENT IF NONE PIVOT IS ON THE LAST ROW $\Rightarrow$ $\boxed{x_4 + x_3 = 0}$

NOTE: ONE LINEAR HOMOG. EQ. FOR A SPACE OF "CODIMENSION" 1, THAT IS A HYPERPLANE (DIMENSION $4-1=3$), WHILE THE RANK IS 3.

Complete one of the following two exercises:

6) Find the distance between the point $P(1,2,-3)$ and the plane $2x - y + 3z = -4$.

DISTANCE FORMULA: PLANE $\vec{X} \cdot \vec{n} = \vec{p} \cdot \vec{n}$ OR $ax+by+cz=d$ AND POINT (x_0, y_0, z_0). DIST $= \dfrac{|ax_0 + by_0 + cz_0 - d|}{\|\vec{n}\|} =$

$$= \frac{|2\cdot 1 + (-1)\cdot 2 + 3\cdot(-3) - (-4)|}{\sqrt{1+4+9}}$$

$$= \frac{5}{\sqrt{14}} \quad \text{OR} \quad \frac{5}{14}\sqrt{14} \quad \text{OR} \quad 1.3363$$

7) Find a unit vector in the direction of the vector $\boldsymbol{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}$. *(Do not approximate, but simplify your answer)*

$$\vec{U} = \text{UNIT VECTOR} = \frac{1}{\|\vec{u}\|}\vec{u}$$

$$\|\vec{u}\| = \sqrt{4+1+9+16} = \sqrt{30}$$

$$\to \vec{U} = \frac{\sqrt{30}}{30}\begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \vec{U} = \begin{bmatrix} \sqrt{30}/15 \\ -\sqrt{30}/30 \\ \sqrt{30}/10 \\ 2\sqrt{30}/15 \end{bmatrix}$$

Complete one of the following three exercises:

8) The augmented matrix is given for a system of equations. If the system is consistent, state which variables (if any) are free and find the general solution, otherwise state that there is no solution.

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 1 & -1 & 3 \\ -2 & 1 & 0 & 2 & 1 & 2 \\ 3 & 2 & 1 & 0 & 0 & 1 \\ 0 & -2 & 3 & 0 & 2 & 1 \end{array}\right] \overset{\text{RREF}}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -30/59 & 38/59 \\ 0 & 1 & 0 & 0 & 19/59 & -28/59 \\ 0 & 0 & 1 & 0 & 52/59 & 1/59 \\ 0 & 0 & 0 & 1 & -10/59 & 111/59 \end{array}\right]$$

FREE VARIABLE

CONSISTENT WITH X_5 FREE VARIABLE

REDUCED SYSTEM:

$$\begin{cases} X_1 - \frac{30}{59} X_5 = 38/59 \\ X_2 + \frac{19}{59} X_5 = -28/59 \\ X_3 + \frac{52}{59} X_5 = 1/59 \\ X_4 - \frac{10}{59} X_5 = 111/59 \\ X_5 = t \end{cases}$$

$$\Rightarrow \text{GEN. SOL}: \vec{X} = \begin{bmatrix} \frac{38}{59} + \frac{30}{59}t \\ -\frac{28}{59} - \frac{19}{59}t \\ \frac{1}{59} - \frac{52}{59}t \\ \frac{111}{59} + \frac{10}{59}t \\ t \end{bmatrix}$$

9) Linda invests $25,000 for one year. Part is invested at 5%, another part at 6%, and the rest at 8%. The total income from all 3 investments is $1600. The total income from the 5% and 6% investments is equal to the income from the 8% investment. Find the amount invested at each rate. (*You must set up a system of linear equations and solve it using the Gauss-Jordan elimination method on the augmented matrix.*)

SET: X_1 INVEST. @ 5% ; X_2 @ 6% ; X_3 @ 8%

TOTAL INVESTED $\rightarrow$ $X_1 + X_2 + X_3 = 25000$

TOTAL RETURN $\rightarrow$ $.05X_1 + .06X_2 + .08X_3 = 1600$

RELATION $\rightarrow$ $.05X_1 + .06X_2 = .08X_3 \rightarrow .05X_1 + .06X_2 - .08X_3 = 0$

$$\text{AUG. MAT.} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25000 \\ .05 & .06 & .08 & 1600 \\ .05 & .06 & -.08 & 0 \end{array}\right] \overset{\text{RREF}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10000 \\ 0 & 1 & 0 & 5000 \\ 0 & 0 & 1 & 10000 \end{array}\right]$$

$X_1 = \$10,000$; $X_2 = \$5,000$; $X_3 = \$10,000$.

10) Use row reduction (showing your computations) in order to compute the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 1 & 5 & 7 \\ 0 & 0 & 3 & 2 \\ -1 & 1 & -4 & 1 \end{bmatrix}. \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_4 \to R_4 + R_1 \end{matrix} \approx \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -3 & 7 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 3 & -5 & 4 \end{bmatrix} R_4 \to R_4 + R_2 \approx$$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -3 & 7 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix} R_4 \to R_4 - \frac{2}{3}R_3 \approx \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -3 & 7 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 11/3 \end{bmatrix} = \text{REF}$$

RANK = # OF NON-ZERO ROWS = 4