Mat321 – Spring 2015 –Exam2

Instructor: Dr. Francesco Strazzullo **Name**________________________

I certify that I did not receive third party help in *completing* this test (sign) ________________________

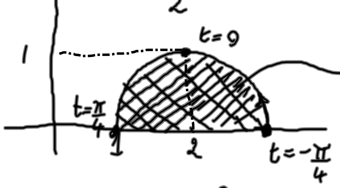
Instructions. Technology and instructor's notes (including the formula sheets from our book) are allowed on this exam. Each problem is worth 10 points. If you use notes or formula sheets, make a reference. When using technology describe which commands (or keys typed) you used or print out your worksheet.

SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Find the area of the region bounded by $x = 2 - \sin(2t)$, $y = \cos(2t)$, $-\frac{\pi}{4} \le t \le \frac{\pi}{4}$, and the x-axis.

HALF CIRCLE OF RADIUS 1

$A = \frac{1}{2}\pi$



$$A = \int_0^1 x\,dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2-\sin(2t))(-2\sin(2t))\,dt =$$

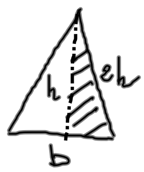
$dy = -2\sin(2t)\,dt$

Integral[(2-sin(2x))(-2*sin(2x)),-pi/4,pi/4]

$\rightarrow \frac{1}{2}\pi \approx 1.571$

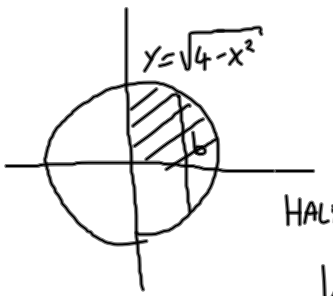
$$\text{ALSO, } A = 2\int_1^2 y\,dx = 2\int_{\frac{\pi}{4}}^{0} \cos(2t)(-2)\cos(2t)\,dt = \frac{1}{2}\pi$$

2. A solid has a circular base of radius 2. Parallel cross-sections perpendicular to the base are isosceles triangles, which the equal sides above the base and as long as twice the height. Find the volume of the solid.



$$(2h)^2 = h^2 + \left(\frac{b}{2}\right)^2 \Rightarrow 3h^2 = \frac{b^2}{4} \Rightarrow$$

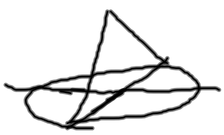
$$\Rightarrow h = \frac{b}{2}\frac{\sqrt{3}}{3} = \frac{b\sqrt{3}}{6}$$



THE VOLUME IS SYMMETRIC, FOUR EQUAL PARTS: $\frac{b}{2} = y$

HALF AREA OF CROSS-SECTIONS $= \frac{1}{4}bh = \frac{b^2\sqrt{3}}{24} = y^2\frac{\sqrt{3}}{6}$

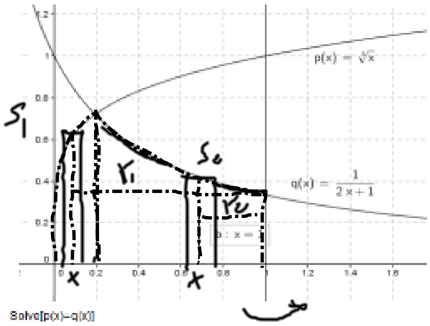
$$V = 4\int_0^2 \frac{\sqrt{3}}{6}(4-x^2)\,dx = \frac{2\sqrt{3}}{3}\left[4x - \frac{x^3}{3}\right]_0^2$$

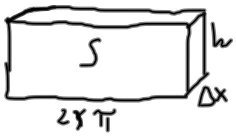


$$= \frac{2\sqrt{3}}{3}\left(8 - \frac{8}{3}\right) = \frac{32\sqrt{3}}{9} \approx 6.158$$

3. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt[5]{x}$, $y = \frac{1}{2x+1}$, the x-axis, and $x = 1$ about the line $x = 1$.

CYLINDRICAL SHELLS, $h_1 = \sqrt[5]{x}$; $h_2 = \frac{1}{2x+1}$





$r_1 = r_2 = 1-x$

$S_1 = 2\pi(1-x)\sqrt[5]{x} \cdot \Delta x$, $0 \le x \le .2$

$S_2 = 2\pi(1-x)/(2x+1) \cdot \Delta x$, $.2 \le x \le 1$

$$V = \int_0^{.2} 2\pi(1-x)\sqrt[5]{x}\, dx + \int_{.2}^{1} 2\pi \frac{(1-x)}{2x+1}\, dx \approx 1.754$$

Solve[p(x)=q(x)]

$\to \{x = -1, x = 0.19405\}$

$\approx .2$

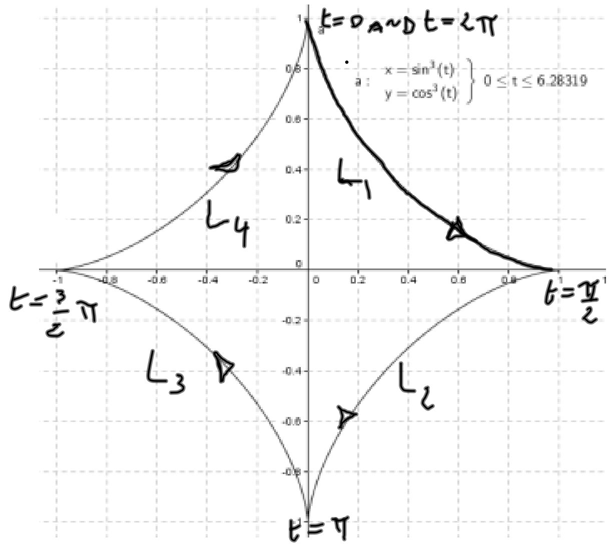
2*pi*(Integral[(1-x)nroot(x, 5),0,.2]+Integral[(1-x)/(2x+1),.2,1])

$\to \frac{3}{2}\pi \ln(3) - \frac{3}{2}\pi \ln\left(\frac{7}{5}\right) + \frac{1}{825}\pi\left(49\sqrt[5]{625} - 660\right)$

3 / 2 π ln(3) - 3 / 2 π ln(7 / 5) + 1 / 825 π (49nroot(625,5) - 660)

= 1.7544

4. Find the length of the curve $x = \sin^3 t$, $y = \cos^3 t$, $0 \le t \le 2\pi$.



$L = L_1 + L_2 + L_3 + L_4 = 4L_1$

BECAUSE OF SYMMETRY

$$L = \int_a^b \sqrt[2]{(x')^2 + (y')^2}\, dt$$

$$L_1 = \int_0^{\frac{\pi}{2}} \sqrt[2]{(x')^2 + (y')^2}\, dt = \int_0^{\frac{\pi}{2}} \frac{3}{2}|\sin(2t)|\, dt = \frac{3}{2}\int_0^{\frac{\pi}{2}} \sin(2t)\, dt$$

$x' = 3\cos t \sin^2 t$

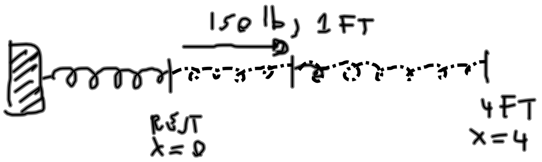
$y' = 3(-\sin t)\cos^2 t$

$\to (x')^2 + (y')^2 = 3^2(\cos t \sin t)^2 (\sin^2 t + \cos^2 t)$, with $\sin^2 t + \cos^2 t = 1$

$= \left(\frac{3}{2}\sin(2t)\right)^2$

$$L_1 = \frac{3}{2}\left[-\frac{1}{2}\cos(2t)\right]_0^{\frac{\pi}{2}} = \frac{3}{4}(-(-1)+1) = \frac{3}{2} \Rightarrow L = 4L_1 = 6$$

5. A spring stretches 1 foot beyond its natural position under a force of 150 pounds. How much work in foot-pounds is done in stretching it 4 feet beyond its natural position?

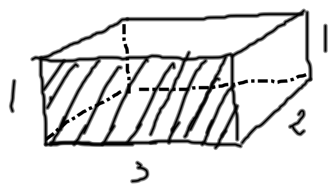


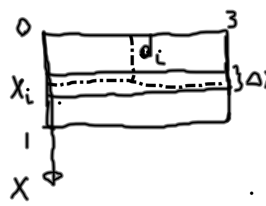
Hooke's Law

$$\left.\begin{array}{l} F = kx \\ 150 = k \cdot 1 \Rightarrow k = 150 \end{array}\right] \to F = 150x$$

$$W = \int_0^4 F\,dx = \int_0^4 150x\,dx = 150\left[\frac{x^2}{2}\right]_0^4 = 1200 \quad \text{FT-lb}$$

6. An aquarium 1 foot high, 2 foot wide, and 3 feet long is filled with water. For simplicity, take the density of water to be 60 lb/ft^3. Find the hydrostatic force in pound on one of the 1 foot by 3 feet sides of the aquarium.





$$\left.\begin{array}{l} d_i = x_i \\ A_i = 3\,\Delta x \\ \delta = 60 \end{array}\right] \to F_i = \delta d_i A_i = 180 x_i \cdot \Delta x$$

$$F \approx \sum_{i=1}^{n} F_i$$

$$F = \int_0^1 180x\,dx = 180\left[\frac{x^2}{2}\right]_0^1 = 90 \quad \text{lb}$$

7. Find c so that the following can serve as the probability density function of a random variable X:

$$f(X) = \begin{cases} cX^2 e^{-5X^3} & \text{if } X \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} c x^2 e^{-5x^3}\,dx = \lim_{b \to -\infty} c \int_0^b x^2 e^u \frac{1}{-15x^2}\,du \Rightarrow$$

$$\boxed{\begin{array}{l} u = -5x^3 \Rightarrow du = -15x^2\,dx \\ x = 0 \Rightarrow u = 0 \quad ; x \to \infty \Rightarrow u \to -\infty \end{array}}$$

$$\Rightarrow 1 = \frac{c}{15} \cdot \lim_{b \to -\infty} \int_b^0 e^u\,du = \frac{c}{15} \lim_{b \to -\infty} \left[e^u\right]_b^0 = \frac{c}{15}\left(1 - \lim_{b \to -\infty} e^b\right)$$

$$\Rightarrow 1 = \frac{c}{15} \cdot (1 - 0) \Rightarrow \frac{c}{15} = 1 \Rightarrow c = 15$$

8. A culture of bacteria is doubling every 3 hours. What is the average population over the first four hours if we assume that the culture initially contained four million organisms?

$$P = P_0 e^{kt} \Rightarrow 2P_0 = P_0 e^{k(3)} \Rightarrow \ln 2 = 3k \Rightarrow k = \frac{\ln 2}{3} \Rightarrow$$

$$P = P_0 e^{\ln 2 \cdot \frac{t}{3}} = P_0 2^{\frac{t}{3}}, \text{ IF } P \text{ IN MILLION, } P_0 = 4 = 2^2 \Rightarrow P = 2^{\frac{t}{3}+2}$$

$$\text{AVERAGE DURING FIRST 4 HOURS} = \frac{1}{4-0}\int_0^4 P\,dt = \frac{1}{4}\int_0^4 2^{\frac{t}{3}+2}\,dt$$

.25*Integral[2^(x/3+2),0,4]

$$\rightarrow \frac{\sqrt[3]{2} \cdot 6 - 3}{\ln(2)}$$

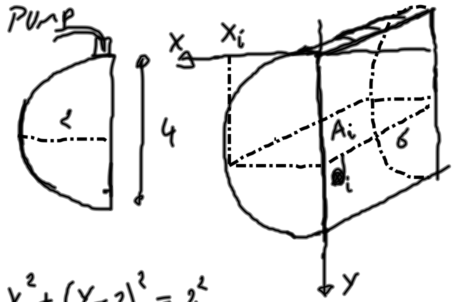
(cbrt(2) 6 - 3) / ln(2)

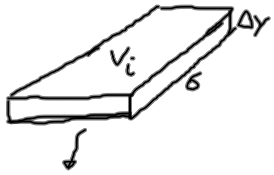
≈ 6.578005,9809

≈ 6.578006

OR ABOUT 6,578,006 ORGANISMS

9. A tank 6 feet long has cross-sections in the shape of a semicircle with radius 2 feet long. Suppose that the tank is filled to a depth of 4 feet with liquid weighing 18 lb/ft³. How much work is required to empty the tank by pumping the liquid over the edge of the tank?





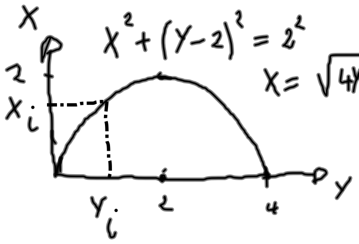
$A_i = 6 \cdot x_i$

$V_i = A_i \cdot \Delta y$

$d_i = y_i$ DEPTH

= DISPLACEMENT

$F_i = \text{WEIGHT}_i = \delta V_i = 18 \cdot 6 x_i \, \Delta y$



$x^2 + (y-2)^2 = 2^2$

$x = \sqrt{4y - y^2}$

$\text{WORK}_i = w_i = d_i F_i = 108 \cdot y_i \cdot x_i \, \Delta y$

$$W = \sum w_i = \int_0^4 108 \cdot y \sqrt{4y - y^2} \, dy \approx$$

108*Integral[x*sqrt(4x-x^2),0,4]

→ 432 π

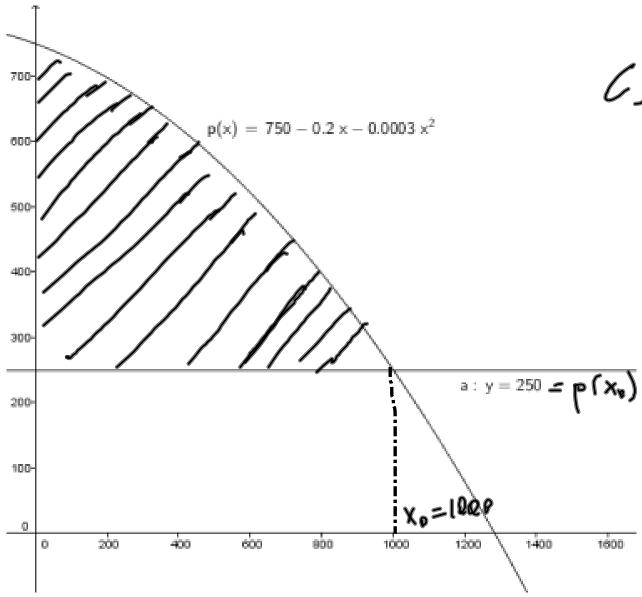
432π

≈ 1357.168026351

FT-lb

Energy	Energy
From	From
1357.168026351	1357.168026351
Foot-Pound	Foot-Pound
To	To
439.4938303783769	1840.072769028188
Calorie	Joule

10. The demand function for producing a certain commodity is given by $p = 750 - 0.2x - 0.0003x^2$. Find the consumer surplus when the sale level is 1000.



$$CS = \int_0^{x_0} p(x) - p(x_0) \, dx$$

$$= \int_0^{1000} 500 - .2x - .0003x^2 \, dx$$

$$= \left[500x - .1x^2 - .0001x^3 \right]_0^{1000}$$

$$= 300{,}000 \text{ (DOLLARS?)}$$