Math 221- Fall 2012 - Test 3

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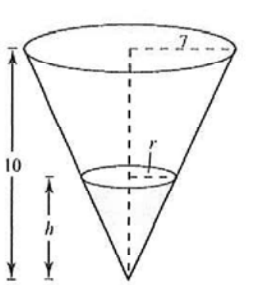
I certify that I did not receive third party help in *completing* this test. (*sign*)

Instructions. You **can not** use a graph to justify your answer. Each problem is worth 10 points.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. A conical tank (with vertex down) is 7 meters across the top and 10 meters deep. Water is flowing into the tank at a constant rate so that the water level is rising at a rate of 5 centimeters per minute. At what rate is the water flowing when it is 2 meters short of the topping line? How long does it take to fill the tank?

"WATER FLOWING" = V = "VOLUME OF WATER"



a) IT IS ASKED TO FIND $\frac{dV}{dt}$ WHEN $10-h=2 \Rightarrow h=8$

AND $\frac{dh}{dt} = 5 \frac{cm}{min} = .05 \frac{m}{min}$.

$$\frac{r}{h} = \frac{7}{10} \Rightarrow r = \frac{7}{10}h$$

$V = \frac{1}{3}\pi r^2 h$ $\longrightarrow$ $V = \frac{1}{3}\pi \left(\frac{7}{10}\right)^2 h^2 h \Rightarrow$

$$\Rightarrow V = \frac{1}{3}\pi \frac{49}{100} h^3 \Rightarrow \frac{dV}{dt} = \left(\frac{1}{\cancel{3}}\pi \frac{49}{100}\right)\cancel{3}h^2 \frac{dh}{dt} \Rightarrow \text{PLUG DATA}$$

$$\frac{dV}{dt} = \pi \frac{49}{100}(8)^2(.05) \approx \boxed{4.93 \ m^3/min}$$

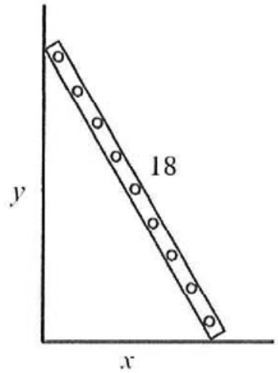
WHEN THE WATER LEVEL IS 8 m, THE WATER IS FLOWING AT A RATE OF 4.93 m^3 PER MINUTE.

b) NOTE THAT $\frac{dV}{dt} = \frac{49\pi}{100} h^2 \frac{dh}{dt}$, THEREFORE EVEN IF WATER IS RISING AT CONSTANT RATE, WITH $h = .05t$, THE RATE OF CHANGE OF VOLUME WITH RESPECT TO TIME IS NOT CONSTANT.

TO GET TO $h = 10$ m WE MUST HAVE $10 = .05t \Rightarrow t = \frac{10}{.05} = 200$ min

IT TAKES 200 MINUTES. TO JUST FILL THE LAST 2 METERS IT WILL TAKE $t = \frac{2}{.05} = 40$ MINUTES.

2. A ladder 18 feet long is leaning against the wall of a house (see scheme). The top of the ladder is dropping at a rate of 2 feet per second. At what rate is the base of the ladder sliding away from the wall when the base is 5 feet from the wall?



"RATE OF CHANGE OF BASE" $= \dfrac{dx}{dt}$

DATA: $x = 5$, $dy/dt = -2$ FT/SEC. (DECREASING)

$$x^2 + y^2 = 18^2 \xrightarrow[\text{DIFFERENTIATION}]{\text{IMPLICIT}} \frac{d}{dt}\left[x^2 + y^2\right] = 0$$

$$\rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \rightarrow$$

$$\rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \quad \left(\underset{\text{NEEDED}}{\overset{\text{NOT}}{\rightarrow}} \frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}\right)$$

WE CAN PLUG THE DATA: $(5)^2 + y^2 = 18^2 \rightarrow y^2 = 299 \rightarrow$

$$\rightarrow y \approx 17.29 \left(= \sqrt{299}\right)$$

$$5\frac{dx}{dt} + \sqrt{299}(-2) = 0 \rightarrow \frac{dx}{dt} = -\frac{-2\sqrt{299}}{5} \approx 6.92 \text{ FT/s}$$

3. Find the absolute extrema and (if any) the relative extrema of the function $y = xe^{-\frac{x}{2}}$ on the closed interval $[\frac{1}{4}, 4]$.

$$y' = (1)e^{-x/2} + x\left(-\frac{1}{2}\right)e^{-x/2} = \left(1 - \frac{x}{2}\right)e^{-x/2} = 0 \to 1 - \frac{x}{2} = 0 \to x = 2$$

CRITICAL #'S: y' IS ALWAYS DEFINED, SO — $e^{-x/2} \neq 0$ NEVER

ABS EXTREMA:

x	1/4	2	4
y	.22	.74	.54

ABS. MIN. (1/4, .22)
ABS. MAX (2, .74) → LOC. MAX.

THE ONLY CRITICAL NUMBER GIVES AN ABS. MAX INSIDE THE INTERVAL AND THUS IT IS ALSO A LOCAL EXTREMA (WE DON'T NEED THE 1ST DER. TEST)

[NOT NEEDED 1ST DERIVATIVE TEST

X	1	3
y'	+	−
y		

CONFIRMING THAT (2, .74) IS A LOCAL MAX]

4. Find the local extrema, the inflection points, and the intervals on which the function

$$f(x) = x^4 - 2x^3 + x - 5$$

is increasing, decreasing, concave up or down.

$$\boxed{f'(x) = 4x^3 - 6x^2 + 1 \quad ; \quad f''(x) = 12x^2 - 12x}$$ BOTH ALWAYS DEFINED

$f'(x) = 0 \to$ BY GRAPH OR ALGEBRA: $f'(.5) = 0$ SYN. DIV.

$$\begin{array}{r|rrrr} 1/2 & 4 & -6 & 0 & 1 \\ & & 2 & -2 & -1 \\ \hline & 4 & -4 & -2 & 0 \end{array}$$ ✓

$$f'(x) = \left(x - \frac{1}{2}\right)2\underbrace{(2x^2 - 2x - 1)}_{=0} \to x = \frac{2 \pm \sqrt{4+8}}{2(2)} = \frac{1 \pm \sqrt{3}}{2} \approx \begin{cases} -.366 \\ 1.366 \end{cases}$$

1ST DER. TEST

		$\frac{1-\sqrt{3}}{2}$		$\frac{1}{2}$		$\frac{1+\sqrt{3}}{2}$	
x	−1		0		1		2
y'	−		+		−		+
y							

$f(x)$ INCREASING ON $\left(\frac{1-\sqrt{3}}{2}, \frac{1}{2}\right) \cup \left(\frac{1+\sqrt{3}}{2}, \infty\right)$

$f(x)$ DECREASING ON $\left(-\infty, \frac{1-\sqrt{3}}{2}\right) \cup \left(\frac{1}{2}, \frac{1+\sqrt{3}}{2}\right)$

LOCAL MAX AT $\left(\frac{1}{2}, f\left(\frac{1}{2}\right) = -\frac{75}{16} = -4.6875\right)$

TWO LOCAL MINIMA AT $\left(\frac{1-\sqrt{3}}{2}, f\left(\frac{1-\sqrt{3}}{2}\right) = -5.25\right)$ AND $\left(\frac{1+\sqrt{3}}{2}, f\left(\frac{1+\sqrt{3}}{2}\right) = 5.25\right)$

$$f''(x) = 12x(x-1) = 0 \begin{cases} x = 0 \\ x = 1 \end{cases}$$

		0		1	
X	−1		.5		2
y''	+		−		+
y'					
y	∪		∩		∪

→ [$f(x)$ CONCAVE UP ON $(-\infty, 0) \cup (1, +\infty)$

$f(x)$ CONCAVE DOWN ON $(0, 1)$

INFLECTION POINTS: $f(0) = f(1) = -5$

(0, −5) HIGHEST RATE

(1, −5) LOWEST RATE]

5. Find the following limits, if defined. Write the known limit or the rule for horizontal asymptote that you use.

(a) (*10 pts*) $\lim_{u\to 0^+} \cos u \ln u =$ $-\infty$

DIRECT SUBSTITUTION AND $\lim_{u\to 0^+} \ln u = -\infty$, $\cos 0 = 1$! $\rightarrow \lim_{u\to 0^+} \cos u \ln u = 1\cdot(-\infty) = -\infty$

(b) (*10 pts*) $\lim_{x\to\infty} (1-3e^{-x})^x =$ 1

DIRECT SUBS $= (1-3(0))^\infty = 1^\infty \rightarrow$ REWRITE TO USE HOSPITAL'S RULE;

$$\lim_{x\to\infty}(1-3e^{-x})^x = \lim_{x\to\infty} e^{\ln(1-3e^{-x})^x} = \lim_{x\to\infty} e^{x\ln(1-3e^{-x})}$$

$$\lim_{x\to\infty} x\ln(1-3e^{-x}) \underset{\text{REWRITE}}{\overset{\infty(0)}{=}} \lim_{x\to\infty}\frac{\ln(1-3e^{-x})}{\frac{1}{x}} \underset{\text{H.R.}}{\overset{\frac{0}{0}}{=}} \lim_{x\to\infty}\frac{\frac{-e^{-x}}{1-3e^{-x}}}{-\frac{1}{x^2}} \Rightarrow$$

$$\Rightarrow \lim_{x\to\infty}(1-3e^{-x})^x = e^{\lim_{x\to\infty} x\ln(1-3e^{-x})} = e^{\lim_{x\to\infty}\frac{x^2e^{-x}}{1-3e^{-x}}} \Rightarrow$$

$$\lim_{x\to\infty} x^2e^{-x} = \lim_{x\to\infty}\frac{x^2}{e^x} \underset{\text{HR}}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty}\frac{2x}{e^x} \underset{\text{HR}}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty}\frac{2}{e^x} = \frac{2}{\infty} = 0$$

$$\Rightarrow \lim_{x\to\infty}(1-3e^{-x})^x = e^{\frac{\lim_{x\to\infty} x^2e^{-x}}{\lim_{x\to\infty} 1-3e^{-x}}} = e^{\frac{0}{1-0}} = e^0 = 1$$

6. Eighty elks are introduced into a game preserve. It is estimated that their population will increase according to the model

$$p(t) = \frac{50}{1+5e^{-t/2}},$$

where t is measured in years. After how many years is the population increasing *most rapidly*? What is the limiting size of this population of elks?

a) "MOST RAPIDLY" $\triangleq$ "MAX FOR p'"

$$p'(t) = 50\,\frac{-5\left(-\frac{1}{2}\right)e^{-t/2}}{\left(1+5e^{-t/2}\right)^2} = 125\,\frac{e^{-t/2}}{\left(1+5e^{-t/2}\right)^2}$$

NOTE: $e^{-t/2}$ AND $1+5e^{-t/2}$ ARE ALWAYS DEFINED AND POSITIVE

$$p''(t) = 125\,\frac{-\frac{1}{2}e^{-t/2}\left(1+5e^{-t/2}\right)^2 - e^{-t/2}\cdot\cancel{2}\left(1+5e^{-t/2}\right)\left(5\left(-\frac{1}{\cancel{2}}\right)e^{-t/2}\right)}{\left(1+5e^{-t/2}\right)^4}$$

$$= 125\,\frac{e^{-t/2}}{\left(1+5e^{-t/2}\right)^3}\left(-\frac{1}{2}-\frac{5}{2}e^{-t/2}+5e^{-t/2}\right) \overset{C.N.'S}{=} 0 \to -\frac{1}{2}+\frac{5}{2}e^{-t/2} = 0 \to$$

$$\to e^{-t/2} = \frac{1}{5} \to -t/2 = \ln\left(\frac{1}{5}\right) \Rightarrow t = 2\ln 5 \approx 3.23,\quad p(2\ln 5) = 25$$

$2\ln 5$

x	1	4
p''	+	−
p'		

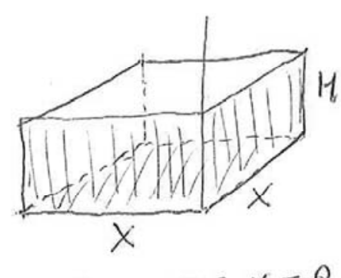
AFTER ABOUT 3 YEARS, WHEN POPULATION IS OF 25 ELKS, IT IS GROWING MOST RAPIDLY.

b) "LIM. SIZE" $= \lim_{t\to\infty} p(t) = \frac{50}{1+5\left(\frac{1}{\infty}\right)} = \frac{50}{1+0} = 50$ ELKS.

7. A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

"AMOUNT OF MATERIAL USED" = SURFACE $= X^2 + 4XH$

$32000 = $ VOLUME $= X^2H \Rightarrow H = \frac{32000}{X^2}$



THEN:

$$S = X^2 + 4X\left(\frac{32000}{X^2}\right) = X^2 + \frac{128000}{X}$$

→ UNDEFINED FOR $X = 0$ OUT OF CONTEXT

$$S' = 2X - \frac{128000}{X^2} = \frac{2X^3-128000}{X^2} = 0 \Rightarrow 2X^3 - 128000 = 0 \Rightarrow$$

$$\Rightarrow \sqrt[3]{X^3} = \sqrt[3]{64000} \Rightarrow X = 40 \text{ cm.}$$

CHECK MIN: $S'' = 2 + 2\frac{(128000)}{X^3} \Rightarrow S''(40) > 0$ $\cup$ ✓

$H = \frac{32000}{(40)^2} = 20$ cm. MINIMIZING DIMENSIONS: 40cm X 40cm X 20cm.

8. Find the most general antiderivative of the function $f(x) = 3e^{3x} - \sin x$, then the antiderivative $F(x)$ that satisfies the condition $F(0) = 1$.

AN ANTIDERIVATIVE OF e^{kx} IS $\dfrac{e^{kx}}{k}$

AN ANTIDERIVATIVE OF $\sin x$ IS $-\cos x$ $\Big] \to F(x) = 3\,\dfrac{e^{3x}}{3} - (-\cos x) + C$

$\Rightarrow \boxed{F(x) = e^{3x} + \cos x + C}$

PLUG DATA: $1 = F(0) = e^{3\cdot 0} + \cos 0 + C \Rightarrow 1 = 1 + 1 + C \Rightarrow C = -1$

PARTICULAR ANTIDERIVATIVE:

$$F(x) = e^{3x} + \cos x - 1$$

9. Check that the function $y = x^2 + \sin(2x)$ is a solution of the second order ODE $y'' - 2xy' + 4y = 2 - 4x\cos(2x)$.

$y' = 2x + 2\cos(2x)$; $y'' = 2 + 2(2)(-\sin(2x)) = 2 - 4\sin(2x)$

L.H.S. $= y'' + 2x\,y' + 4y = 2 - 4\sin(2x) - 2x\left(2x + 2\cos(2x)\right) +$

$+ 4\left(x^2 + \sin(2x)\right) = 2 - 4\sin(2x) - 4x^2 - 4x\cos(2x) + 4x^2 + 4\sin(2x)$

$= 2 - 4x\cos(2x) =$ R.H.S ✓