Math 221- Fall 2014 - Test 3 - Part 1/2

Instructor: Dr. Francesco Strazzullo My Name KEY

I certify that I did not receive third party help in *completing* this test. (*sign*)

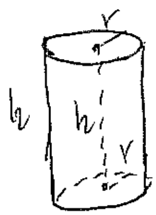
Instructions. You **can not** use a graph to justify your answer. Each problem is worth 10 points.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Let V be the volume of a cylinder having height h and radius r, and assume that h and r vary with time. When the height is 4 in. and is increasing at 0.3 in./s, the radius is 2.5 in. and is decreasing at 0.2in./s.

   (a) How fast is the volume changing at that instant?

   (b) Is the volume increasing or decreasing at that instant?

(a)



$V = \pi r^2 h$

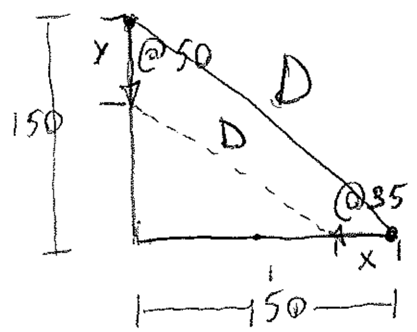
GIVEN: $h = 4$, $h' = +.3$
$r = 2.5$, $r' = -.2$

ASKING FOR V' ⇒

$$\Rightarrow V' = \pi\left(2rr'h + r^2h'\right)$$
$$= \pi\left(2(2.5)(-.2)\cdot 4 + (2.5)^2(.3)\right) = \pi\left(-\frac{17}{8}\right)$$
$$= -\frac{17}{8}\pi \approx -6.68 \text{ in}^3/\text{s}$$

(b) AT THE TIME CONSIDERED, THE VOLUME IS DECREASING AT A RATE OF ABOUT 6.7 CUBIC INCHES PER SECOND.

2. Two cars are each 150 miles away from the town of Waleska, one directly to the north and the other directly to the east. The car to the north (on GA 140) is heading toward the town at 50 miles per hour, while the one to the east (on GA 108) is heading toward the town at 35 miles per hour. How fast are the cars approaching each other?



$$D^2 = (150 - x)^2 + (150 - y)^2 \quad (*)$$

ASKED FOR D':

GIVEN: $x' = 35$ CONSTANT $\Rightarrow x = x't = 35t$

ALSO $y' = 50$ AND $y = y't = 50t$.

$D' = \frac{d}{dt}[D] \Rightarrow$ APPLY TO THE EQUATION $(*)$

$$2D \cdot D' = 2(150 - x)(-x') + 2(150 - y)(-y') \Rightarrow$$

$$\Rightarrow D' = -\frac{(150 - x)x' + (150 - y)y'}{\sqrt{(150 - x)^2 + (150 - y)^2}}$$

PLUG GIVEN VALUES AND EXPRESSIONS:

$$D' = -\frac{(150 - 35t) \cdot 35 + (150 - 50t)50}{\sqrt{(150 - 35t)^2 + (150 - 50t)^2}} = \frac{-5^2(210 - 49t + 300 - 100t)}{\sqrt{5^2(900 - 140t + 49t^2 + 900 - 600t + 100t^2)}}$$

$$D' = \frac{-5(510 - 149t)}{\sqrt{1800 - 740t + 149t^2}} \text{ mph}$$

IN PARTICULAR, AT THE BEGINNING, FOR $t = 0$ ONE HAS:

$$D'(0) = -\frac{5(510)}{\sqrt{1800}} = -\frac{85}{\sqrt{2}} \approx -60 \text{ mph}$$

3. Find the absolute extrema and (if any) the relative extrema of the function $y = 1.5 + e^{x^2-5x+4}$ on the closed interval $[1, 3]$.

$Y = f(x)$ IS CONTINUOUS ON $[1,3]$: IT MUST HAVE BOTH ABS MAX AND MIN

CRIT. #: $f'(x) = (2x-5)\,e^{x^2-5x+4}$ (ALWAYS DEFINED)

$f'(x) = 0 \iff 2x-5=0 \iff x = 5/2 = 2.5$

ABS EXTREMA:

x	1	2.5	3
$f(x)$	2.5	≈ 1.6	≈ 1.64

$\Rightarrow$ ABS MAX: $(1, 2.5)$; ABS MIN: $(2.5, 1.6)$

A CRITICAL POINT THAT IS AN ABS. EXTR. IS ALSO A RELATIVE ONE, THEN $(2.5, 1.6)$ IS ALSO A RELATIVE MIN.

4. Use differentiation to find the local extrema, the inflection points, and the intervals on which the function

$$f(x) = x^3 + 2x^2 - 3x + 4$$

is increasing, decreasing, concave up or down.

$f'(x) = 3x^2 + 4x - 3$ (NEVER UNDEFINED)

$f''(x) = 6x + 4$ (NEVER UNDEFINED)

•) LOCAL EXTREMA $\Rightarrow$ CRIT. # $\Rightarrow f'(x) = 0 \Rightarrow x = \frac{-4 \pm \sqrt{16+36}}{6} = \frac{-2 \pm \sqrt{13}}{3}$

1ST DERIVATIVE TEST

x	-2	$(-2-\sqrt{13})/3$	0	$(-2+\sqrt{13})/3$	1
SIGN OF $f'(x)$	+		−		+
$f(x)$ INCR.	↗		↘		↗

$f(x)$ IS:

INCREASING ON $\left(-\infty, \frac{-2-\sqrt{13}}{3}\right) \cup \left(\frac{-2+\sqrt{13}}{3}, \infty\right)$

DECREASING ON $\left(\frac{-2-\sqrt{13}}{3}, \frac{-2+\sqrt{13}}{3}\right)$

LOCAL EXT. REL MAX: $\left(\frac{-2-\sqrt{13}}{3}, 10.07\right)$ AND REL MIN $\left(\frac{-2+\sqrt{13}}{3}, 3.12\right)$

••) CONCAVITY. $f''(x) = 0 \iff 6x + 4 = 0 \iff x = -\frac{2}{3}$

x	-1	$-2/3$	0
SIGN OF $f''(x)$	−		+
CONC. OF $f(x)$	$\frown$		$\smile$

$f(x)$ IS:

CONCAVE UP ON $(-2/3, \infty)$ AND CONC. DOWN ON $(-\infty, -\frac{2}{3})$

ONE INFLECTION POINT: $(-\frac{2}{3}, 6.59)$

5. Find the following limits, if defined. Write the known limit or the rule for horizontal asymptotes that you use.

(a) *(10 pts)* $\lim_{u\to 0^+} \frac{\sin u}{\ln u} \underset{\text{SUB}}{=} \frac{0}{-\infty}$ $\Rightarrow$ REWRITE AFTER MANIPULATION:

$$\lim_{u\to 0^+} \frac{\sin u}{\ln u} = \lim_{u \to 0^+} \sin u \cdot \frac{1}{\ln u} = 0 \cdot \frac{1}{-\infty} = 0 \cdot 0 = 0$$

(b) *(10 pts)* $\lim_{x\to 0^-} (1-\cos(3x))^{\frac{1}{x}} \underset{\text{SUBS}}{=} 0^{-\infty}$ $\Rightarrow$ APPLY H-R. AFTER MANIPULATION.

WITH EXPONENTIAL UNDETERMINATE USE $y = e^{\ln y}$ IF DEFINED

$$\lim_{x\to 0^-} (1-\cos(3x))^{\frac{1}{x}} = \lim_{x\to 0^-} e^{\ln(1-\cos(3x))^{\frac{1}{x}}} = e^{\lim_{x\to 0^-} \frac{1}{x} \ln(1-\cos(3x))} \underset{\text{SUBS}}{=} e^{-\infty \cdot (-\infty)} = e^{+\infty}$$

$$= \lim_{y \to +\infty} e^{y} = +\infty$$

(c) (*10 pts*) $\lim_{x\to 2} \frac{x-2}{3-\sqrt{x^3+1}} = \frac{0}{0}$ SUBS   H.R. OR ALGEBRAIC SIMPLIFICATION

$$= \lim_{x\to 2} \frac{(x-2)(3+\sqrt{x^3+1})}{9-(x^3+1)} = \lim_{x\to 2} \frac{(x-2)(3+\sqrt{x^3+1})}{8-x^3} =$$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(3+\sqrt{x^3+1})}{-\cancel{(x-2)}(4+2x+x^2)} = -\frac{3+\sqrt{8+1}}{4+4+4} = \frac{-6}{12} = -\frac{1}{2}$$

6. Your GPA depends on the hours t you spend on completing your weekly homework, according to the model

$$G(t) = \frac{4}{0.88+e^{3-0.5t}}. = 4\,(.88+e^{3-.5t})^{-1}$$

At what level of weekly work is your GPA increasing *most rapidly*? What is the limiting size of your GPA?

$$G'(t) = 4(-1)(.88+e^{3-.5t})^{-2}(-.5e^{3-.5t})$$

$$= -2\,\frac{e^{3-.5t}}{(.88+e^{3-.5t})^2} \Rightarrow$$

$$\Rightarrow G''(t) = -2\,\frac{-.5e^{3-.5t}(.88+e^{3-.5t})^2 - e^{3-.5t}(2)(-.5)(.88+e^{3-.5t})}{(.88+e^{3-.5t})^4}$$

$$= (+1)\,\frac{e^{3-.5t}\cdot(.88+e^{3-.5t})(.88+e^{3-.5t}-2)}{(.88+e^{3-.5t})^4} \Rightarrow$$

$$\Rightarrow G''(t) = \frac{e^{3-.5t}}{(.88+e^{3-.5t})^3}(e^{3-.5t}-1.12)$$ NEVER UNDEFINED. THEN $G''(t)=0 \Rightarrow$

$$\Rightarrow e^{3-.5t}-1.12=0 \Rightarrow 3-.5t=\ln(1.12) \Rightarrow t=2(3-\ln(1.12)) \approx 5.77$$

NOTE: THE FACTOR OF $G''(t)$ $\frac{e^{3-.5t}}{(.88+e^{3-.5t})^3} > 0$.

$2(3-\ln(1.12))$

t	5	6
SIGN $G''(t)$	+	−
$G'(t)$ INCR.		

REL MAX FOR G'

STUDYING ABOUT 5 HOURS AND 46 MINUTES THE GPA GROWS MOST RAPIDLY.

LIMITING SIZE $= \lim_{t\to\infty} G(t) = \frac{4}{.88+e^{-\infty}} = \frac{4}{.88+0} = \frac{50}{11} \approx 4.55$

7. A company has a cost function $C(x) = 150 - 60x + x^2$ and demand function $p(x) = 100 - x^2$, where x is the number of units made. How many units should it make to maximize its profit? (NO MORE THAN 200 UNITS)

REVENUE = QUANTITY · PRICE = $x \cdot p(x) = 100x - x^3$

PROFIT = REVENUE − COST $\overset{\text{DEMAND}}{=} -x^3 - x^2 + 160x - 150$

$P'(x) = -3x^2 - 2x + 160 = 0 \Rightarrow x = \dfrac{2 \pm \sqrt{4 + 1920}}{-6} = \dfrac{-1 \pm \sqrt{481}}{3} \Rightarrow$

CRIT. #'S

$\Rightarrow x \approx 6.98, \quad x \approx -7.64$ (TO REJECT)

NOT NEEDED

CHECK IT GIVES MAX, BY 2ND DERIVATIVE TEST:

$P''(x) = -6x - 2 \Rightarrow P''\left(\dfrac{-1 + \sqrt{481}}{3}\right) < 0 \Rightarrow$ REL. MAX.

IMPOSE THE CLOSED INTERVAL CONDITIONS: $0 \leq x \leq 200$

x	0	$\frac{-1+\sqrt{481}}{3}$	200
$P(x)$	−150	578.01	−80K \$

ABS. MAX

TO CHOOSE THE APPROXIMATION:

$x = 6 \to P(6) = \$558$

$x = 7 \to P(7) = \$578 \to$ MAXIMUM PROFIT OF \$578

WHEN PRODUCING 7 UNITS.

8. Find all value(s) of c (if any) that satisfy the conclusion of the Mean Value Theorem for the function $f(x) = e^{x^2-3x+1}$ on the interval $[-1, 3]$.

MVT: THERE IS A SOLUTION $x = c$, $a < c < b$, TO THE EQUATION

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

$$(2x-3)\, e^{x^2-3x+1} = \frac{e - e^5}{3-(-1)} \overset{\text{GRAPHICALLY}}{\Longrightarrow} x \approx -.3768$$

9. Consider the function $f(x) = \cos(2x)$ and use Newton's method to find

(a) the iterative formula for x_{n+1}, and

(b) the third approximation x_3 of the solution of the equation $f(x) = \frac{1}{4}$ with the initial approximation $x_1 = \frac{\pi}{6}$.

(a) $$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \overset{*}{=} x_n - \frac{\cos(2x_n) - \frac{1}{4}}{-2\sin(2x_n)}$$

(b)

$$x_2 = \frac{\pi}{6} + \frac{\cos(\frac{\pi}{3}) - \frac{1}{4}}{2\sin(\frac{\pi}{3})} \approx .667936$$

$$x_3 = x_2 + \frac{\cos(2x_2) - \frac{1}{4}}{2\sin(2x_2)} \approx .659077$$

$$x_4 \approx .659058$$

* NOTE: APPROXIMATING SOLUTIONS OF $F(x) = 0$:
HERE THE EQUATION IS $f(x) = \frac{1}{4}$, THUS $f(x) - \frac{1}{4} = 0$ AND
WE USE $F(x) = f(x) - \frac{1}{4} = \cos(2x) - \frac{1}{4}$

$$F'(x) = -2\sin(2x)$$

Math 221- Fall 2014 - Test 3 - Part 2/2

Instructor: Dr. Francesco Strazzullo Name V(3Y

Instructions. You **can not** use a graph to justify your answer. This problem is worth 10 points, therefore your Test 3 will be graded out of 120 points.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

Find (if any) the absolute and the relative extrema of the function $y = x^4 + 2x^3 - x^2 - 3x - 5$.

$$y' = 4x^3 + 6x^2 - 2x - 3 = 2x^2(2x+3) - (2x+3) = (2x^2-1)(2x+3)$$

$$y'' = 12x^2 + 12x - 2 \quad (\text{NOT NEEDED})$$

CRITICAL #'S : 1) y' ALWAYS DEFINED.

2) $y' = 0 \Rightarrow \begin{cases} 2x^2 - 1 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2} \approx \pm .7 \\ 2x + 3 = 0 \Rightarrow x = -3/2 \end{cases}$

1ST DERIVATIVE TEST IS FASTER THEN EVALUATING y'' (2ND DER. TEST)

	$-3/2$		$-\sqrt{2}/2$		$\sqrt{2}/2$	
x						
	-2		-1	0	1	

x	-2	-1	0	1
SIGN $f'(x)$	(+)(−) −	(+)(+) +	(−)(+) −	(+)(+) +
INC. DECR. $f(x)$	↘	↗	↘	↗
RELATIVE	MIN	MAX	MIN	

REL. MIN AT $\left(-\frac{3}{2}, -4.438\right)$
AND AT $\left(\frac{\sqrt{2}}{2}, -6.664\right)$

REL. MAX AT $\left(-\frac{\sqrt{2}}{2}, -3.836\right)$

Y IS AN EVEN POLYNOMIAL WITH POSITIVE LEADING COEFFICIENT, THEREFORE IT IS GOING TO ∞ AS $x \to \pm\infty$ AND Y MUST HAVE AN ABSOLUTE MINIMUM, WHICH MUST BE THE SMALLEST OF ITS RELATIVE MINIMA.

ABS. MIN $\left(\frac{\sqrt{2}}{2}, -6.664\right)$