Mat321 – Spring 2013 -Exam3

Instructor: Dr. Francesco Strazzullo **Name** KEY

I certify that I did not receive third party help in *completing* this test (sign) ____________

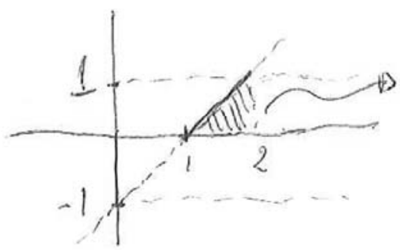
Instructions. Technology and instructor's notes (including the formula sheets from our book) are allowed on this exam. Each problem is worth 10 points. If you use notes or formula sheets, make a reference. When using technology describe which commands (or keys typed) you used or print out your worksheet.

SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Find the area of the region bounded by $x = 1 + \sin t$, $y = \sin t$, $0 \le t \le \pi$, and the x-axis.

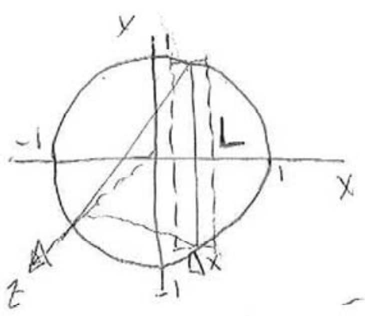
$$\begin{cases} y = \sin t \\ x = 1 + \sin t \end{cases} \quad 0 \le t \le \pi \xrightarrow{\text{PLUG } y = \sin t \text{ IN } x = 1 + \sin t} \boxed{x = 1 + y, \quad 0 \le y \le 1, \quad x \ge 0}$$

$$\to y = x - 1$$

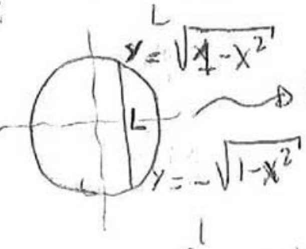


$$A = \int_1^2 x - 1\, dx = \text{AREA OF TRIANGLE} = \frac{1}{2}(1)(1) = \boxed{\frac{1}{2}}$$

2. A solid has a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.



$$\left.\begin{array}{l} h = \frac{\sqrt{3}}{2} L \\ A = \frac{1}{2} h L \end{array}\right] \to A = \frac{\sqrt{3}}{4} L^2$$



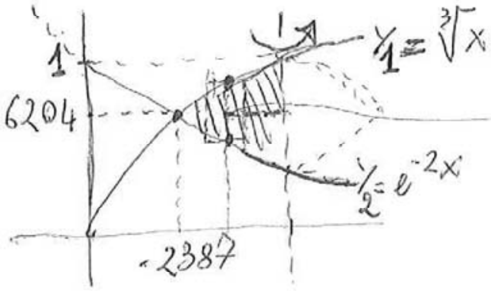
$$\to L = 2|y| = 2\sqrt{1-x^2}$$

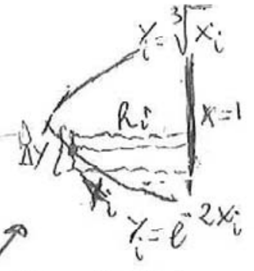
$$\to A = \frac{\sqrt{3}}{4}\left(2\sqrt{1-x^2}\right)^2$$

$$\to A = \sqrt{3}\,(1 - x^2) \Rightarrow V = \int_{-1}^{1} A(x)\, dx = \int_{-1}^{1} \sqrt{3}\,(1 - x^2)\, dx = 2\sqrt{3}\int_0^1 (1 - x^2)\, dx$$

$$= 2\sqrt{3}\left[x - \frac{x^3}{3}\right]_0^1 = \boxed{\frac{4}{3}\sqrt{3}} \approx 2.3094$$

3. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt[3]{x}$, $y = e^{-2x}$ and $x = 1$ about the line $x = 1$.



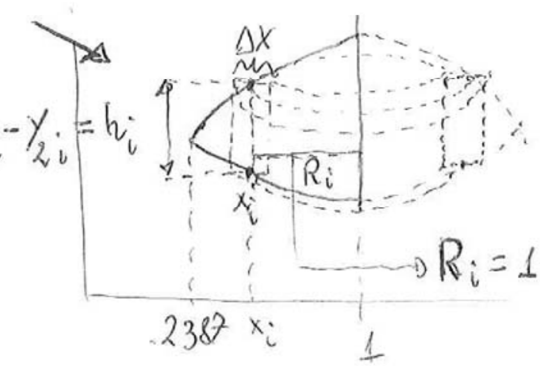


V_i OF DISK $= \pi R_i^2 \cdot h_i$

$h_i = \Delta y$

$$R_i = 1 - x_i \begin{cases} 1 - y_i^3, & .6204 \le y \le 1 \\ 1 - \left(-\frac{1}{2}\ln y\right), & 0 \le y \le .6204 \end{cases}$$

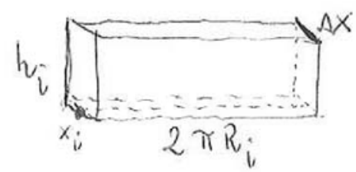
IT WILL BE "EASIER" WITH CYLINDRICAL SHELLS



$$V = \int_{.2387}^{1} 2\pi R h \, dx =$$

$$= 2\pi \int_{.2387}^{1} (1-x)\left(\sqrt[3]{x} - e^{-2x}\right) dx \approx 0.6951$$

TI OR CAS



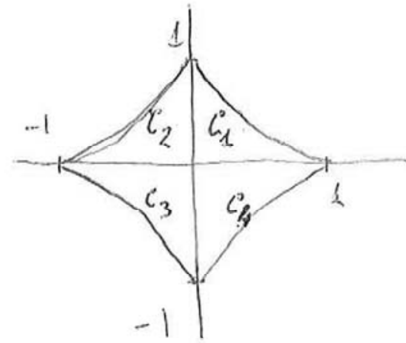
CYL. SHELL

$V_i = 2\pi R_i h_i \Delta x$

4. Find the length of the curve $x = \sin^3 t$, $y = \cos^3 t$, $0 \le t \le 2\pi$.

$L_i = |c_i|$ LENGTH OF THE i-TH BRANCH.

BECAUSE $f(t) = t^3$ IS SYMMETRIC WITH RESPECT TO THE ORIGIN AND BECAUSE $\cos(t) = \cos(-t)$, $L_1 = L_2 = L_3 = L_4$ (OUR CURVE IS SYMMETRIC WITH RESPECT THE X-AXIS AND THE Y-AXIS).



$$L = \int_0^{2\pi} \sqrt{[\sin^3(t)']^2 + [\cos^3(t)']^2}\, dt = 4L_1 = 4\int_0^{\frac{\pi}{2}} \sqrt{[\sin^3(t)']^2 + [\cos^3(t)']^2}\, dt$$

$$= 4 \int_0^{\frac{\pi}{2}} \sqrt{[3\sin^2(t)\cos(t)]^2 + [3\cos^2(t)(-\sin t)]^2}\, dt =$$

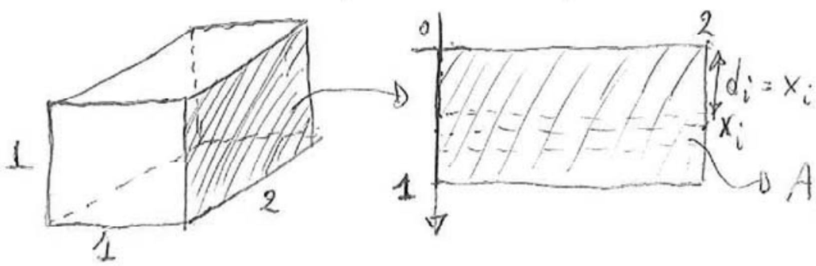
$$= 4\int_0^{\frac{\pi}{2}} 3 \sin t \cos t \sqrt{\sin^2 t + \cos^2 t}\, dt = 12 \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin(2t)\, dt = 6\left[-\frac{1}{2}\cos(2t)\right]_0^{\frac{\pi}{2}} = 6$$

5. A spring stretches 1 foot beyond its natural position under a force of 100 pounds. How much work in foot-pounds is done in stretching it 3 feet beyond its natural position?

Hook's: $F = KX \to 100 = K(1) \to K = 100$

$$W = \int_0^3 F\,dx = \int_0^3 100X\,dx = \frac{100}{2}\left[X^2\right]_0^3 = 50(9) = 450 \text{ lb·ft}$$

6. An aquarium 1 foot high, 1 foot wide, and 2 feet long is filled with water. For simplicity, take the density of water to be 60 lb/ft^3. Find the hydrostatic force in pound on one of the 1 foot by 2 foot sides of the aquarium.



$A_i = 2\,\Delta X$

$\delta = 60 \text{ lb/ft}^3$

$F_i = \delta d_i A_i = 60 \cdot x_i (2) \Delta X$

$$F = \int_0^1 60X(2)\,dX = 120\int_0^1 x\,dx = 120\left[\frac{X^2}{2}\right]_0^1 = 60 \text{ lb (pounds)}$$

7. Find c so that the following can serve as the probability density function of a random variable X:

$$f(X) = \begin{cases} cXe^{-4X^2} & \text{if } X \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

PROB. DENSITY FUNCTION IF $\int_{-\infty}^{\infty} f(x)\,dX = 1.$

$$1 = \int_{-\infty}^{\infty} f(x)\,dX = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dX = 0 + \lim_{t \to \infty} \int_0^t cXe^{-4X^2}\,dX$$

$$\int cXe^{-4X^2}\,dX = c\int Xe^u \frac{du}{-8X} = -\frac{c}{8}\int e^u\,du = -\frac{c}{8}e^u + K = -\frac{c}{8}e^{-4X^2} + K$$

$$u = -4X^2 \Rightarrow dX = du/u' = (-8X)^{-1}du$$

$$\Rightarrow 1 = \lim_{t \to \infty}\left[-\frac{c}{8}e^{-4X^2}\right]_0^t = -\frac{c}{8}\lim_{t \to \infty}\left[e^{-4t^2} - 1\right] = -\frac{c}{8}(0-1) \Rightarrow c = 8$$

8. A culture of bacteria is doubling every hour. What is the average population over the first two hours if we assume that the culture initially contained two million organisms?

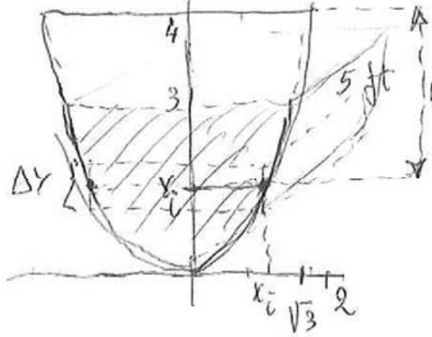
EXPONENTIAL GROWTH: $P = P_0 e^{Kt} \xrightarrow[\text{EVERY HOUR}]{\text{DOUBLING}} 2 = e^{K \cdot 1} \Rightarrow K = \ln 2 \Rightarrow$

$$\Rightarrow P = P_0 e^{(\ln 2)t} = P_0 2^t \Rightarrow P = 2 \cdot 2^t = 2^{t+1}$$ MILLION OF ORGANISMS

AVERAGE DURING FIRST TWO HOURS $(t=0, t=2)$ $= \frac{1}{2-0}\int_0^2 P\,dt = \frac{1}{2}\int_0^2 2^{t+1}\,dt = 1/2\left[\frac{2^{t+1}}{\ln 2}\right]_0^2 =$

$$= \frac{1}{2\ln 2}\left(2^3 - 2\right) = \frac{3}{\ln 2} \approx 4.328085$$ MILLIONS

9. A tank 5 feet long has cross-sections in the shape of a parabola $y = x^2$, for $-2 \le x \le 2$ (where x and y are in feet). Suppose that the tank is filled to a depth of 3 feet with liquid weighing 15 lb/ft^3. How much work is required to empty the tank by pumping the liquid over the edge of the tank?



VOLUME OF i-TH "SLICE" $= 2X_i \cdot (5) \cdot \Delta Y = 10\sqrt{Y_i}\,\Delta Y$

d_i = i-TH DISPLACEMENT $= 4 - Y_i$

i-TH WEIGHT $= V_i\,\delta = (10\sqrt{Y_i}\,\Delta Y)(15)$

$\rightarrow$ i-TH WORK $= W_i = (4 - Y_i)\,150\sqrt{Y_i}\,\Delta Y \rightarrow$

$$\Rightarrow W = \int_0^3 150(4-y)\sqrt{y}\,dy = 150\int_0^3 4y^{\frac{1}{2}} - y^{3/2}\,dy =$$

$$= 150\left[4\frac{y^{3/2}}{3/2} - \frac{y^{5/2}}{5/2}\right]_0^3 = 300\left[y^{3/2}\left(\frac{4}{3} - \frac{1}{5}y\right)\right]_0^3 = 220\sqrt{27}$$

$$= 660\sqrt{3} \approx 1143.15 \text{ ft}\cdot\text{lb}$$

10. The demand function for producing a certain commodity is given by $p = 1000 - 0.1x - 0.0001x^2$. Find the consumer surplus when the sale level is 500.

SALES LEVEL $= 500 \Rightarrow X = 500 \Rightarrow p(500) = 925$

CONSUMER SURPLUS $= \displaystyle\int_0^{500} p - 925\,dx = \int_0^{500} 75 - 0.1X - .0001X^2\,dx$

$= 20833\frac{1}{3}$

≈ 20833.33 DOLLARS