MAT 421 – Exam3 – Take Home – FALL 2017

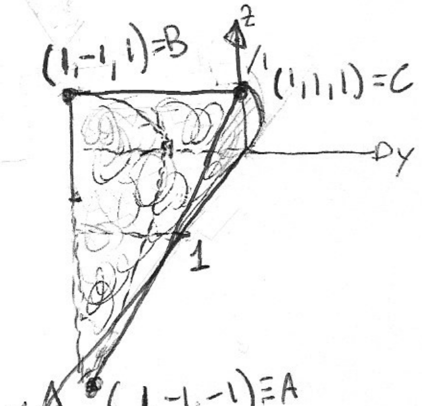
Instructor: Dr. Francesco Strazzullo **Name** ________ Vεx

I certify that I did not receive third party help in *completing* this test (sign)________________________

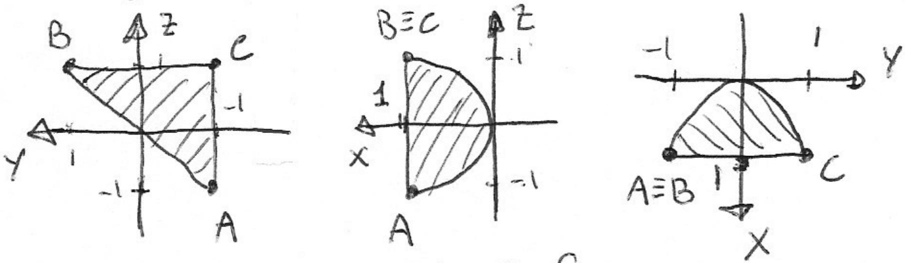
Instructions. This is an open book test. Each exercise is worth 10 points. When approximating, use four decimal places. You cannot use a CAS to justify your answers, but only to perform computations.

SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Find $\iiint_S 2x + y - z\, dV$, where S is the solid bounded by the surface $x = z^2$ and the planes $x = 1$, $y = -1$, $z = 1$, and $z = y$.



$$S = \{-1 \le y \le 1,\ 1 \le z \le y,\ z^2 \le x \le 1\}$$



SECTIONS OF S

$$\iiint_S 2x+y-z\,dV = \int_{-1}^{1} dy \int_{1}^{y} dz \int_{z^2}^{1} 2x+y-z\,dx =$$

$$= \int_{-1}^{1} dy \int_{1}^{y} \left[x^2+yx-zx\right]_{z^2}^{1} dz = \int_{-1}^{1} dy \int_{1}^{y} 1+y-z-z^4-yz^2+z^3\,dz =$$

$$= \int_{-1}^{1} \left[-\frac{z^5}{5}+\frac{z^4}{4}-y\frac{z^3}{3}-\frac{z^2}{2}+yz+z\right]_{1}^{y} dy$$

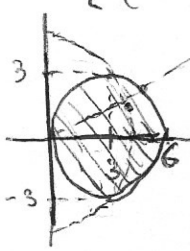
$$= \int_{-1}^{1} -\frac{y^5}{5}+\frac{y^4}{4}-\frac{y^4}{3}-\frac{y^2}{2}+y^2+y+\frac{1}{5}-\frac{1}{4}+\frac{y}{3}+\frac{1}{2}-y-1\,dy$$

$$= \int_{-1}^{1} -\frac{y^5}{5}-\frac{y^4}{12}+\frac{y^2}{2}+\frac{1}{3}y-\frac{11}{20}\,dy = \left[-\frac{y^6}{30}-\frac{y^5}{60}+\frac{y^3}{6}+\frac{y^2}{6}-\frac{11y}{20}\right]_{-1}^{1} =$$

$$= -\frac{4}{15}-\frac{8}{15} = -\frac{12}{15} = -\frac{4}{5} = -.8$$

2. Find the center of mass of the lamina $R = \{x^2 - 6x + y^2 \le 0\}$ with density distribution $\rho(x, y) = x^2 + y^2$.

$R = \{(x-3)^2 + y^2 \le 9\}$; $x^2 + y^2 \le 6x \Rightarrow r^2 \le 6r\cos\theta \Rightarrow r \le 6\cos\theta \Rightarrow$



$\Rightarrow R^* = \{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 6\cos\theta\}$

$$M = \iint_R \rho(x,y)\,dA = \iint_{R^*} r^2 \cdot r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{6\cos\theta} r^3\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{6^4}{4}\cos^4\theta\,d\theta$$

$$= \frac{243}{2}\pi$$

SYMMETRY W.R.T. X-AXIS AND ρ EVEN $\Rightarrow \bar{y} = 0$

$$M_y = \iint_R x\,\rho(x,y)\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{6\cos\theta} r\cos\theta \cdot r^3\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \left[\frac{r^5}{5}\right]_0^{6\cos\theta} d\theta =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{6^5}{5}\cos^6\theta\,d\theta = 486\pi \quad \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{486\pi}{\frac{243}{2}\pi} = 4$$

CENTER OF MASS $= (4, 0)$

3. The joint probability density function of the continuous random variables X and Y is

$$f(x,y) = \begin{cases} 0.07\, e^{-(0.1x+0.7y)} & \text{if } x \ge 0, y \ge 0, \\ 0 & \text{otherwise} \end{cases}$$

Setting up the integrals and using technology, compute the probabilities:

(a) $P(X < 3,\ Y < 2)$

(b) $P(X > -1,\ Y > 4)$

$$P(X<3, Y<2) = \int_{-\infty}^{2} dy \int_{-\infty}^{3} f(x,y)\,dx = \int_0^2 dy \int_0^3 .07\, e^{-(.1x+.7y)}\,dx =$$

$$= .07 \int_0^2 e^{-.7y}\,dy \cdot \int_0^3 e^{-.1x}\,dx = .07 \cdot \left[-\frac{e^{-.7y}}{.7}\right]_0^2 \cdot \left[\frac{-e^{-.1x}}{.1}\right]_0^3 =$$

$$= (e^{-1.4} - 1)(e^{-.3} - 1) \approx .1953 = 19{,}53\%$$

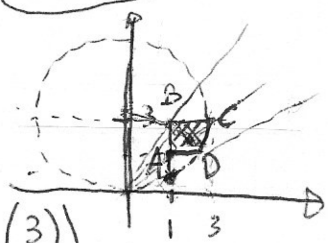
$$P(X>-1, Y>4) = \lim_{b\to\infty} .07 \int_0^b e^{-.1x}\,dx \cdot \int_4^b e^{-.7y}\,dy = \lim_{b\to\infty} \left[-e^{-.1t}\right]_0^b \cdot \left[-e^{-.7t}\right]_4^b =$$

$$= -\lim_{b\to\infty} \left(e^{-.7b} - e^{-2.8}\right) = e^{-2.8} \approx 0.0608 = 6.08\%$$

4.Find $\int_1^3 \int_1^{\sqrt{6y-y^2}} x^2 + y^2 \; dx\, dy$ by converting to polar coordinates.

$R = \{ 1 \le y \le 3,\ 1 \le x \le \sqrt{6y - y^2} \}$, $1 \le x^2 \le 6y - y^2 \Rightarrow$

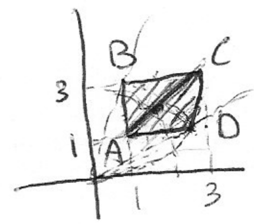
$\Rightarrow \begin{cases} x^2 + (y-3)^2 \le 9 \\ x \ge 1 \end{cases}$



$D = (\sqrt{5}, 1) \equiv (r, \theta) = (\sqrt{6}, \operatorname{atan}(\frac{1}{\sqrt{5}})) \approx (2.45, .42)$

$B = (1,3) \equiv (r, \theta) = (\sqrt{10}, \operatorname{atan}(3)) \approx (3.16, 1.25)$

$C = (3,3) \equiv (r, \theta) = (3\sqrt{2}, \frac{\pi}{4})$



$A = (1,1) \equiv (r, \theta) = (1, \frac{\pi}{4})$

$R^* = \{ 1 \le r\sin\theta \le 3,\ r\cos\theta \ge 1,\ r \le 6\sin\theta \}$

$= \overset{\Delta}{ABC} \cup \overset{\frown}{ACD}$

$x^2 \le 6y - y^2 \Rightarrow x^2 + y^2 \le 6y \Rightarrow$
$\Rightarrow r \le 6\sin\theta$

$\overset{\frown}{ACD} = \{ \operatorname{atan}(\frac{1}{\sqrt{5}}) \le \theta \le \frac{\pi}{4},\ \frac{1}{\sin\theta} \le r \le 6\sin\theta \}$ $\quad \overline{AD} \equiv r\sin\theta = 1 \Rightarrow r = \frac{1}{\sin\theta}$

$\overset{\Delta}{ABC} = \{ \frac{\pi}{4} \le \theta \le \operatorname{atan}(3),\ \frac{1}{\cos\theta} \le r \le \frac{3}{\sin\theta} \}$ $\quad \overline{AB} \equiv r\cos\theta = 1 \Rightarrow r = \frac{1}{\cos\theta}$

$\overline{BC} \equiv r\sin\theta = 3 \Rightarrow r = \frac{3}{\sin\theta}$

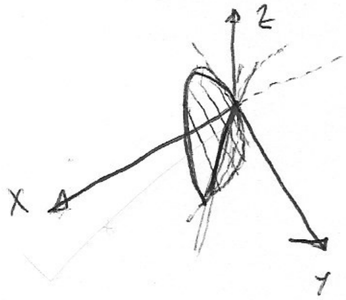
$$\iint_R x^2 + y^2 \, dA = \iint_{R^*} r^2 \cdot r \, dr\, d\theta = \iint_{ACD} r^3 \, dr\, d\theta + \iint_{ABC} r^3 \, dr\, d\theta =$$

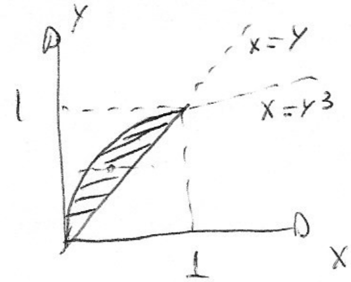
$$= \int_{\operatorname{atan}(\frac{1}{\sqrt{5}})}^{\frac{\pi}{4}} \int_{\frac{1}{\sin\theta}}^{6\sin\theta} r^3 \, dr\, d\theta + \int_{\frac{\pi}{4}}^{\operatorname{atan}(3)} \int_{\frac{1}{\cos\theta}}^{3/\sin\theta} r^3 \, dr\, d\theta = \frac{1}{4} \int_{\operatorname{atan}(\frac{1}{\sqrt{5}})}^{\frac{\pi}{4}} 6^4 \sin^4\theta - \frac{1}{\sin^4\theta} \, d\theta +$$

$$+ \frac{1}{4} \int_{\frac{\pi}{4}}^{\operatorname{atan}(3)} \frac{3^4}{\sin^4\theta} - \frac{1}{\cos^4\theta} \, d\theta \approx \frac{1}{4}(49.9404 + 69.3333) \approx 29.82$$

5. Use a double integral to find the volume of the solid under the surface $z = 2x - y^2$, and above the region bounded by $x = y$ and $y^3 = x$.

$\to z \geq 0$





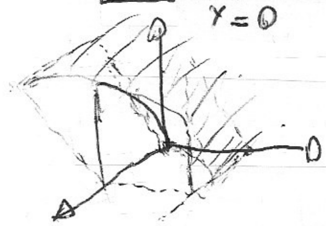
$$R = \{0 \leq y \leq 1,\ y^3 \leq x \leq y\}$$

$$V = \iint_R 2x - y^2 \, dA = \int_0^1 dy \int_{y^3}^{y} 2x - y^2 \, dx = \int_0^1 \left[x^2 - y^2 x\right]_{y^3}^{y} dy =$$

$$= \int_0^1 y^2 - y^3 - y^6 + y^5 \, dy = \left[\frac{y^3}{3} - \frac{y^4}{4} - \frac{y^7}{7} + \frac{y^6}{6}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} - \frac{1}{7} + \frac{1}{6} = \frac{3}{28}$$

6. Find the volume of the solid enclosed by the paraboloid $x = 3y^2 + z^2$ and the panes $z = 0$, $y = 3$, $z = -y$, and $x = 0$. TYPO $Y=0$



ASSUME FIRST OCTANT

$$E = \{3 \le y \le \infty,\ -y \le z \le 0,\ 0 \le x \le 3y^2 + z^2\}$$

IT IS UNBOUNDED THEN $V = \iiint_E dV = \infty$

$$V = \iiint_E dV = \int_3^{\infty} dy \int_{-y}^{0} dz \int_0^{3y^2+z^2} dx = \int_3^{\infty} dy \int_{-y}^{0} 3y^2 + z^2\, dz$$

$$= \int_3^{\infty} \left[3y^2 z + \frac{z^3}{3}\right]_{-y}^{0} dy = \int_3^{\infty} 3y^3 + \frac{y^3}{3}\, dy =$$

$$= \frac{10}{3}\left[\frac{y^4}{4}\right]_3^{\infty} = \infty.$$

IF YOU SET $0 \le y \le 3$, THEN $V = \frac{10}{3}\left[\frac{y^4}{4}\right]_0^3 = \frac{135}{2} = 67.5$

7. Find the Jacobian of the transformation $x = 3v - u^2$, $y = 3u + w^2$, $z = w + 3v^2$.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} -2u & 3 & 0 \\ 3 & 0 & 2w \\ 0 & 6v & 1 \end{vmatrix} \underset{\text{LAST ROW}}{=}$$

$$= -6v \begin{vmatrix} -2u & 0 \\ 3 & 2w \end{vmatrix} + \begin{vmatrix} -2u & 3 \\ 3 & 0 \end{vmatrix} =$$

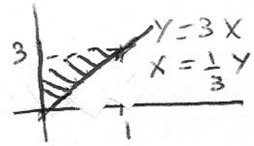
$$= -6v(-4uw) + (0 - 9) = 24uvw - 9$$

MAT 421 – Exam3 – Part 2/2 – Fall 2017

Instructor: Dr. Francesco Strazzullo **Name** KEY

Instructions. This exercise is worth 10 points.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).
Evaluate the iterated integral $\int_0^1 \int_3^{3x} e^{y^2-1}\,dy\,dx$.

$$R=\left\{0\le x\le 1,\ 3x\le y\le 3\right\}$$



$$\int_0^1\int_3^{3x} e^{y^2-1}\,dy\,dx=\int_0^1\left(-\int_{3x}^{3} e^{y^2-1}\,dy\right)dx=-\iint_R e^{y^2-1}\,dA$$

$$=-\int_0^3\int_0^{\frac{1}{3}y} e^{y^2-1}\,dx\,dy=-\int_0^3\left(\tfrac{1}{3}y-0\right)e^{y^2-1}\,dy=*$$

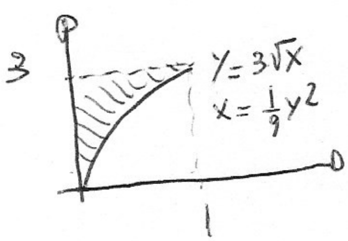
$$R=\left\{0\le y\le 3;\ 0\le x\le \tfrac{1}{3}y\right\}$$

$$*=-\int_0^3 \tfrac{1}{3}y\,e^{y^2-1}\,dy=-\frac{1}{3}\int_0^3\frac{1}{2}\cdot 2y\,e^{y^2-1}\,dy=-\frac{1}{6}\left[e^{y^2-1}\right]_0^3=-\frac{1}{6}\left(e^{8}-e^{-1}\right)$$

$$\approx -496.8$$

$\int e^u u'\,dt=e^u+c$; HERE $u=y^2-1 \Rightarrow u'=2y$

HONOR: $$\int_0^1\int_3^{3\sqrt{x}} e^{y^2-1}\,dy\,dx=\int_0^1\left(-\int_{3\sqrt{x}}^{3} e^{y^2-1}\,dy\right)dx=-\iint_R e^{y^2-1}\,dA=*$$



$$R=\left\{0\le x\le 1,\ 3\sqrt{x}\le y\le 3\right\}=\left\{0\le y\le 3,\ 0\le x\le \tfrac{1}{9}y^2\right\}$$

$$*=-\int_0^3\int_0^{\frac{1}{9}y^2} e^{y^2-1}\,dx\,dy=-\int_0^3\left(\tfrac{1}{9}y^2-0\right)e^{y^2-1}\,dy=$$

$$=-\frac{1}{9}\int_0^3 y^2 e^{y^2-1}\,dy=-\frac{1}{9}\Big(\underbrace{\left[\tfrac{1}{2}y e^{y^2-1}\right]_0^3-\int_0^3\tfrac{1}{2}e^{y^2-1}\,dy}_{uv-\int v\,du}\Big)\approx-\frac{1}{9}\left(\tfrac{3}{2}e^{8}-265.71\right)$$

TECH.

$$\approx -467.3$$

BY PARTS: $u=y \rightarrow du=dy$
$dv=y e^{y^2-1} \Rightarrow v=\frac{1}{2}\int 2y e^{y^2-1}\,dy=\frac{1}{2}e^{y^2-1}$