

College Algebra

Systems of Equations and Inequalities with Matrices addendum to Lesson2

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Objectives

These notes concern topics covered in our textbook, at sections 2.3, 7.1, 7.2, and 8.1. Screenshots of the TI-84 are generated using the softwares IrfanView4.25 and TI-SmartView2.0. We have the following objectives.

- Translate a collection of verbal statement into a system of linear equations.
- Solve a system of two linear equations in two variables by using a graphing utility.
- Solve a system of two linear equations in two variables by the algebraic methods of substitution and elimination.
- Solve a system of linear equations in three variables by elimination.
- Define matrices and the augmented matrix associated to a system of linear equations.
- Define the Reduced Row Echelon Form (or RREF) of a matrix.
- Compute the RREF of a matrix by using a calculator.
- Solve a system of linear equations by computing the RREF of its augmented matrix.
- Solve a system of nonlinear inequality by using a graphing utility.
- Apply systems of linear equations and systems of inequalities to solve problems and answer questions.

1 Systems of Two Linear Equations in Two Variables

Suppose you are given the following problem:

Example 1.1 The total annual consumption of salmon in the United States is given by $y = 5.7x + 815.4$ and that the total annual consumption of tuna in the United States is given by $y = 1.5x + 865.3$, where consumption is measured in millions of pounds and x is the number of years since 1980. Find the year in which U.S. consumption of salmon reaches U.S. consumption of tuna.

In this case we see that the two variables x and y are related by two distinct linear models, whose equations are given to us. The variable x represents the number of years since 1980: for instance, $x = 3$ in 1983 while to year 1995 it corresponds $x = 1995 - 1980$, that is $x = 15$. The variable y represents the consumption in millions of pounds of two distinct goods, so that in general at every given year (i.e. x -value) the consumption of salmon and that of tuna (i.e. the corresponding y -values) are distinct. For instance, in 1980, 1981, and 1995 (i.e. for $x = 0$, $x = 1$, and $x = 15$) the consumptions (in millions of pounds) are computed in Table 1.

These computations are completed with our TI-83/84 in Figure 1.

Year	x	Salmon	Tuna
1980	0	$y = 5.7(0) + 815.4 = 815.4$	$y = 1.5(0) + 865.3 = 865.3$
1981	1	$y = 5.7(1) + 815.4 = 821.1$	$y = 1.5(1) + 865.3 = 866.8$
1995	15	$y = 5.7(15) + 815.4 = 900.9$	$y = 1.5(15) + 865.3 = 887.8$

Table 1:

We notice that for a while salmon consumption is smaller than tuna consumption, but in 1995 the first is bigger.

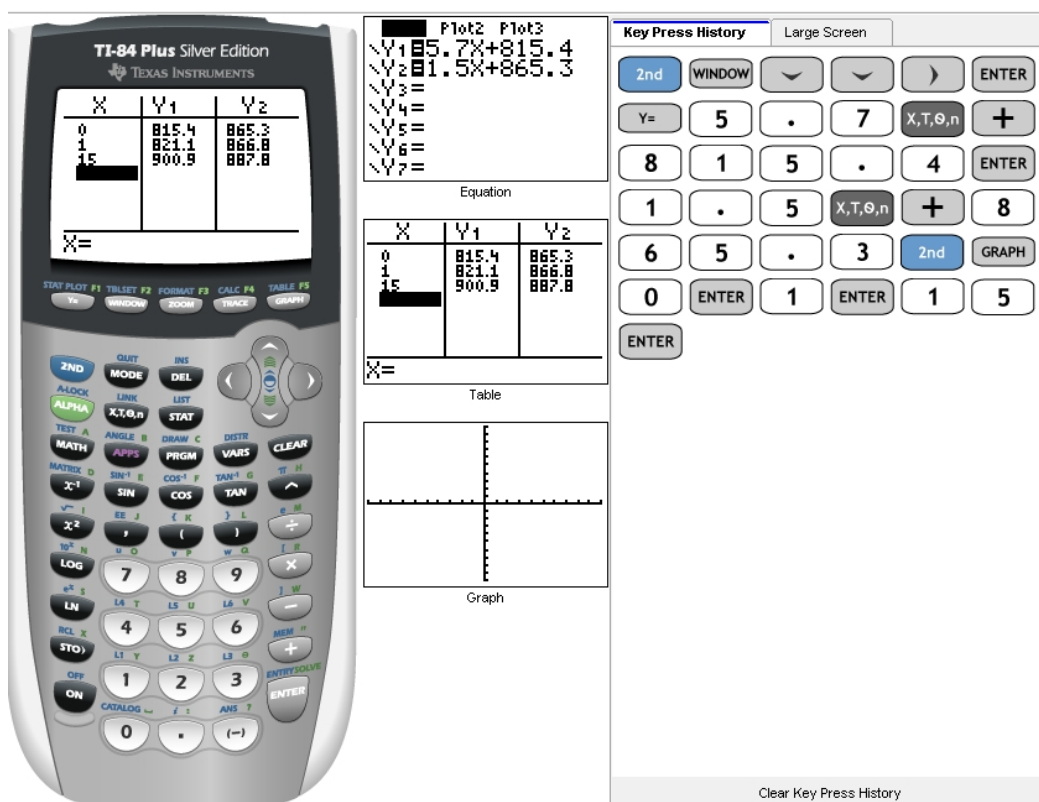


Figure 1:

Therefore there must have been a time when these two consumptions matched, that is the “ y -values” were the same. When we consider this possibility, we try to **solve the equations simultaneously**. This is symbolically noted by writing the **system of two linear equations in two variables**

$$\begin{cases} y = 5.7x + 815.4 \\ y = 1.5x + 865.3 \end{cases} \quad (1)$$

Definition 1.2 A **system of equations** is just a collection of equations. A **solution of a system of equations** is a **solution common to all the equations in the system**, while the collection of all solutions is called **the solution (set)**. One calls **linear** a system whose equations are all linear equations.

In Example 1.1 we are asked to find a solution to the “ 2×2 -system” (1). There are basically two ways to do this: graphically and algebraically.

1.1 Solving Graphically

A system of two linear equations in two variables has an easy graphical interpretation. In a system like (1)

$$\begin{cases} y = 5.7x + 815.4 \\ y = 1.5x + 865.3 \end{cases} \quad (2)$$

each equation represents a straight line in the (x, y) -plane: finding a common solution means finding a “shared” point, or **point of intersection**.

Remark 1.3 Because two straight lines can either

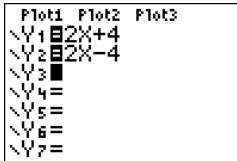
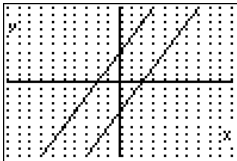
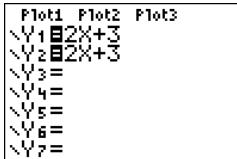
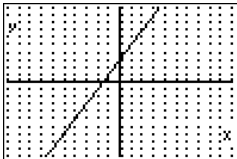
1. be **parallel** without none common point, or
2. be **coincident**, that is parallel with every point shared, or
3. be **incident lines**, that is they intersect in a unique point,

we infer that the solution set of a system of two linear equations in two variables (“ 2×2 ”-system) can contain either

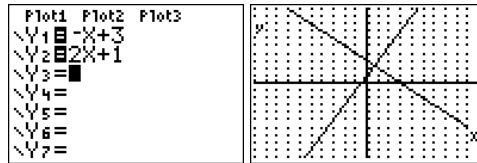
1. none point (“inconsistent system” or **no solution**), or
2. infinitely many points (“consistent system” with **many solutions**), or
3. a unique point (“consistent system” with **one solution**).

Note: to plug the lines in our graphing calculator we must solve all the equations for y in terms of x .

Here there are some examples.

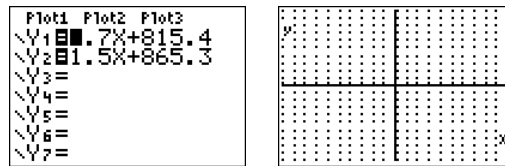
1.	$\begin{cases} 4x - 2y = -8 \\ y = 2x - 4 \end{cases}$	rewritten as	$\begin{cases} y = 2x + 4 \\ y = 2x - 4 \end{cases}$			no solution.
2.	$\begin{cases} 4x - 2y = -6 \\ y = 2x + 3 \end{cases}$	rewritten as	$\begin{cases} y = 2x + 3 \\ y = 2x + 3 \end{cases}$			many solutions.

3. $\begin{cases} x + y = +3 \\ y = 2x + 1 \end{cases}$ rewritten as $\begin{cases} y = -x + 3 \\ y = 2x + 1 \end{cases}$



one solution.

In order to solve graphically a “ 2×2 ”-system we must enter the two models in the “ y -window” $Y=$, as done earlier in Figure 1. Once we click “graph” GRAPH we will likely see an empty viewing window.



That’s because we must setup the correct dimensions of our viewing window, which is by default set to $[-10, 10] \times [-10, 10]$. Here we use the standard notation, where the first interval $[-10, 10]$ is the x -range, that is $X_{\min} = -10$ and $X_{\max} = 10$, while the second interval $[-10, 10]$ is the y -range, that is $Y_{\min} = -10$ and $Y_{\max} = 10$. The context will tell us what feasible window dimensions we should use. For instance, x and y represent positive quantities, therefore we should use $X_{\min} = 0$ and $Y_{\min} \geq 0$. From what we said earlier, after Table 1, we should consider $x \leq 15$, that is $X_{\max} = 15$. From our equations (1) and the outputs shown in Table 1 we actually see that the smallest and the largest y -values (for $0 \leq x \leq 15$) are respectively 815.4 and 900.9: in order to see these values not at the edges of our viewing window, we could set $Y_{\min} = 800$ and $Y_{\max} = 920$. Finally, because y changes of 120 units and x changes of 15 units, we could set $Y_{\text{sc1}} = 12$ so that there will be a vertical thick every 12 units. In Figure 2 we can see how to set the viewing window dimensions to $[0, 15] \times [800, 920]$ with $Y_{\text{sc1}} = 12$.

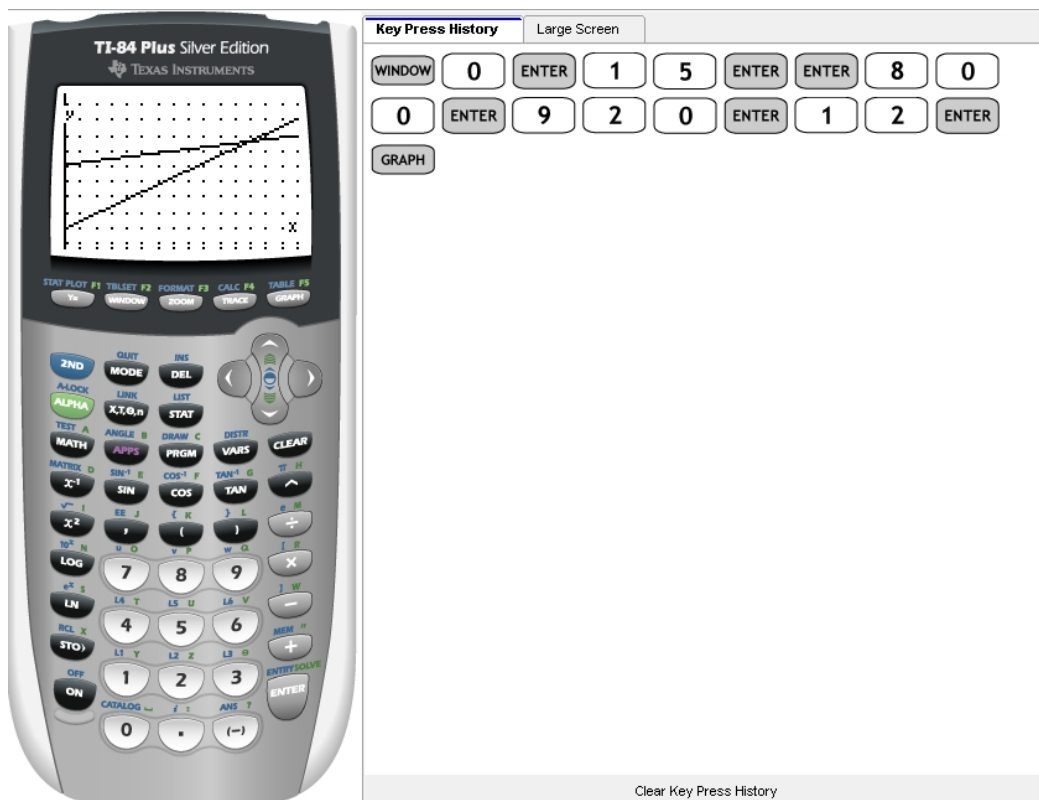
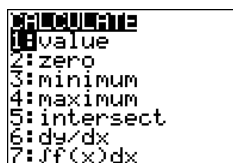



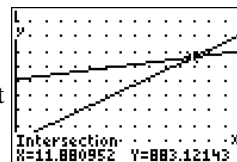
Figure 2: Adjusting the viewing window

As we suspected, we see that the two lines intersect at one point, which is the solution of our system (1). To find the coordinates of this point we use a predefined application of our TI, in the following way.

1. Type   to enter the “calculus window”



2. Select “intersect”, which by default should be at “5”.
3. Now we should be in the “graphing window” where we were able to see the solution: the “trace” should be blinking on the first line, as indicated at the top left corner. At the bottom left corner there should be written “First curve?”, indicating that the calculator is waiting for our selection. Since there are going to be only two curves (straight lines) on our screen, we just have to take the trace as close as possible to the intersection and then we just need to type  three times.



4. On the screen there should be the coordinates of the intersection point

We can conclude that the solution of our system (1) is approximately the point (11.88, 883.12), that is the approximated pair of values $x = 11.88$ and $y = 883.12$. In our Example 1.1 we were asked to provide the year when the y 's were the same, therefore we need to approximate the year to the closest integer and say that the answer is the year 1992 (i.e. $x = 12$). A better approximation would be to say that the consumptions are the same toward the end of year 1991.

1.2 Solving Algebraically

With graphing methods we obtain (usually) approximated answers. For instance, the fraction

$$\frac{13}{17} \approx 0.76470588235294117647058823529412 \dots = 0.76470588\overline{23529411}$$

would be approximated as 0.7647059 in our viewing window and we would hardly be able to recover the exact value $\frac{13}{17}$. In order to compute the exact symbolic answers we need to use algebraic methods. We will consider two of them, the substitution and the elimination method, and we will apply them to our Example 1.1. Finally, we will end this section with the cumulative Example 1.6

1.2.1 Substitution

The substitution method consists in applying the following procedure:

Definition 1.4 (Substitution Method)

1. Solve one equation for one variable;
2. Plug (1) in the other equation, which becomes an equation in one variable;
3. Solve (2);
4. Plug (3) in (1) and evaluate.

Let's apply this method to Example 1.1, that is the system

$$\begin{aligned} EQ_1 & \begin{cases} y = 5.7x + 815.4 \end{cases} \\ EQ_2 & \begin{cases} y = 1.5x + 865.3 \end{cases} \end{aligned} \quad (3)$$

Here we start:

1. Solve one equation for one variable: they are both already solved for y , so we choose the first equation (or EQ_1);
2. Plug (1) in the other equation (or EQ_2): we get

$$5.7x + 815.4 = 1.5x + 865.3.$$

3. Solve (2):

$$5.7x + 815.4 = 1.5x + 865.3 \quad \text{add } -1.5x - 815.4 \text{ on both sides}$$

$$4.2x = 49.9 \quad \text{divide both sides by 4.2}$$

$$x = \frac{499}{42} = 11 + \frac{37}{42} \approx 11.8809523.$$

4. Plug (3) in (1) and evaluate: that is, plug $x = \frac{499}{42}$ into $y = 5.7x + 815.4$

$$y = 5.7 \left(\frac{499}{42} \right) + 815.4 = \frac{123637}{140} = 883 + \frac{17}{140} \approx 883.12142857.$$

The algebraic solution just obtained is well approximated by the graphic solution.

1.2.2 Elimination

Let's recall that a linear equation in two variables (say x and y) is said to be written *in standard form* if all the terms in x and y are written on the left hand side (in alphabetic order) and the known term is written on the right, that is

$$Ax + By = C \quad (A, B, \text{ and } C \text{ are constants.})$$

For instance the equation $3x - 5y = -\frac{2}{5}$ is in standard form, while the equation $5x - 2y = 3x + 4$ is not, because of the $3x$ term on the right. We can rewrite any linear equation in standard form: for example add $-3x$ on both sides and simplify the x -term to obtain

$$5x - 2y = 3x + 4 \Leftrightarrow 2x - 2y = 4 \quad (4)$$

Just notice that a standard form is not unique, for example we can divide by 2 all terms in the last equation (4) and obtain

$$5x - 2y = 3x + 4 \Leftrightarrow 2x - 2y = 4 \Leftrightarrow x - y = 2. \quad (5)$$

Those in (5) are distinct equations of the same straight line $y = x - 2$.

Now that we agree on "notation" we can outline the elimination method:

Definition 1.5 (Elimination Method)

1. Write all the equations in standard form (using the same order for the variables);
2. Multiply one or both equations by numbers such that the numerical coefficients of one variable become opposite numbers.
3. Add the equations obtained at (2), left-hand-side with left-hand-side and so on;
4. At (3) we obtain one equation in one variable and we solve it;
5. Plug (4) in any of the initial equations and solve for the remaining variable.

Let's apply this method to our system (3).

1. Write all the equations in standard form (using the same order for the variables): we bring the x -terms to the left and we obtain

$$\begin{cases} EQ_1 & -5.7x + y = 815.4 \\ EQ_2 & -1.5x + y = 865.3 \end{cases}$$

2. Multiply one or both equations by numbers such that the numerical coefficients of one variable become *opposite numbers*: the y -terms look easy and we can just multiply EQ_1 by -1

$$\begin{cases} EQ_1 & 5.7x - y = -815.4 \\ EQ_2 & -1.5x + y = 865.3 \end{cases}$$

3. Add the equations obtained at (2), left-hand-side with left-hand-side and so on: the y -terms cancel out (y is eliminated) and we get the equation in x

$$4.2x = 49.9$$

4. At (3) we obtain one equation in one variable and we solve it:

$$x = \frac{499}{42} = 11 + \frac{37}{42} \approx 11.8809523.$$

5. Plug (4) in any of the initial equations and solve for the remaining variable: we can plug $x = \frac{499}{42}$ into EQ_1 and obtain

$$y = 5.7 \left(\frac{499}{42} \right) + 815.4 = \frac{123637}{140} = 883 + \frac{17}{140} \approx 883.12142857.$$

1.3 Recap Example

Example 1.6 A chemist has two solutions that contain different concentrations of a certain acid. One is a 3% concentration and the other is an 10% concentration. How many cubic centimeters (cc or cm^3) of each should the chemist mix to obtain 60 cc of a solution with an acid concentration of 5%?

[**Solution 1.6**] In this example our first goal is to translate the descriptions into “math language”. Let’s read the all problem once or twice. First we must figure out what it asked for. From the question “How many cubic centimeters of each ...” we realize that we need to find two quantities: x cc’s of the solution at 4% and y cm^3 of that at 10%. Now we are ready to translate the descriptions and data into (linear) equations. The unknown quantities x and y must provide 60 cc of a (final) solution at 5%, therefore we know that

- “ x and y together make 60 cc”, that is

$$x + y = 60 \tag{6}$$

- the final solution must be such that 5% of the 60 cc is acid, that is the total acid in cubic centimeters is

$$5\% \times 60 = \frac{5}{100}(60) = .05(60) = 3;$$

- each solution contributes to the acid total according to its concentration: the one at 4% gives $.04x$ cc’s of acid and the other one gives $.10y$ cc’s of acid, therefore the required amount of acid is obtained if x and y satisfy the equation

$$.04x + .1y = .05(60). \tag{7}$$

Note that equation (7) is obtained by “sticking the rates” (or concentrations in this case) to the “total-equation” (6). Now we need to solve the 2×2 -system

$$\begin{cases} EQ_1 & x + y = 60 \\ EQ_2 & .04x + .1y = 3 \end{cases} \tag{8}$$

Elimination Our equations are already in standard form. We can multiply EQ_1 by $-.04$ and obtain

$$\begin{cases} EQ_1 & -.04x - .04y = -2.4 \\ EQ_2 & .04x + .1y = 3 \end{cases}$$

Now we add side-by-side, in order to eliminate x , and we obtain the equation

$$.06y = .6$$

that can be solve for y if we divide by .06. Therefore

$$y = 10.$$

Now we plug this in the original EQ_1 of (8), then we solve for x

$$x + 10 = 60 \Leftrightarrow x = 60 - 10 = 50.$$

Answer 1.6 Therefore the chemist needs to mix 50 cc at 4% and 10 cc at 10%.

Substitution We start by solving for y our EQ_1 in (8):

$$y = 60 - x. \quad (9)$$

Then we plug this in EQ_2 and we solve for x :

$$.04x + .1(60 - x) = 3 \Leftrightarrow .04x + 6 - .1x = 3 \Leftrightarrow -.06x = 3 - 6 \Leftrightarrow -.06x = -3 \Leftrightarrow x = \frac{-3}{-.06} = 50.$$

Now we can plug it in (9) and we obtain

$$y = 60 - 50 = 10.$$

As expected, our answer agrees with Answer 1.6.

Graphically Finally, let's try the graphic method. We need to solve both equations for y

$$\begin{cases} EQ_1 & y = 60 - x \\ EQ_2 & .1y = 3 - .04x \Leftrightarrow y = 30 - .4x \end{cases}$$

We plug these in our TI-84 by typing $\boxed{Y=}$ and we adjust the window so that we see the intersection point. Afterward, we type $\boxed{2nd} \boxed{TRACE}$, select "intersect", and finally type \boxed{ENTER} three times.

2 Systems of Three Linear Equations in Three Variables

The standard form of a linear equation in three variables, say x , y , and z , is an equation

$$Ax + By + Cz = D \quad (A, B, C, \text{ and } D \text{ are constants.})$$

For example $3x - \pi y + \frac{2}{3}z = \sqrt{5}$ is a linear equation in three variables. A solution is a triple $(\bar{x}, \bar{y}, \bar{z})$ that, when plugged in the equation, makes this a true equality. For instance in this last example we can see that $(\frac{\sqrt{5}}{3}, \frac{2}{3}, \pi)$ is a solution. Therefore, in order to represent the solution set of a linear equation in three variables we need "three dimensions", that is a 3D coordinate system. The solution set is a flat surface, called plane. A system of three linear equations in three variables is

$$\begin{cases} EQ_1 & A_1x + B_1y + C_1z = D_1 \\ EQ_2 & A_2x + B_2y + C_2z = D_2 \\ EQ_3 & A_3x + B_3y + C_3z = D_3 \end{cases}$$

and the solution set is represented by the intersection of three planes in the 3D space.