Math 310 - Spring 2016 - Test 2 - Take Home

Instructor: Dr. Francesco Strazzullo My Name____________________

I certify that I did not receive third party help in completing this test. (*sign*)____________________

Instructions. SHOW YOUR WORK neatly, please. Each exercise is worth 10 points. If using a result from the book, make a reference to it.

1. Let $H \leq G$, $x \in G$, and $y \in G$. Prove that the relation defined in G by

$$x \,\mathcal{L}_H\, y \Leftrightarrow y^{-1}x \in H,$$

is an equivalence.

ONE CAN PROVE THAT " $x \mathcal{L}_H y \Leftrightarrow xH = yH$ " BY USING 6.3, THEN USE 6.4 AND 1.25 TO PROVE THAT $\mathcal{L}_H$ IS AN EQUIVALANCE.

DIRECTLY, PROVE THAT $\mathcal{L}_H$ IS REFLEXIVE, SYMMETRIC, AND TRANSITIVE.

REFL. PROVE $x \in G \Rightarrow x \mathcal{L}_H x$.

$\forall x \in G, \quad x^{-1}x = e \in H$ (BECAUSE $H \leq G$) $\Rightarrow x \mathcal{L}_H x$ ✓

SYM. PROVE $x \mathcal{L}_H y \Rightarrow y \mathcal{L}_H x$.

$x \mathcal{L}_H y \Rightarrow y^{-1}x \in H \overset{H \leq G}{\Longrightarrow} H \ni (y^{-1}x)^{-1} = x^{-1}(y^{-1})^{-1} = x^{-1}y \Rightarrow y \mathcal{L}_H x$ ✓

TRANS. PROVE $x \mathcal{L}_H y$ AND $y \mathcal{L}_H z \Rightarrow x \mathcal{L}_H z$.

$$\left.\begin{array}{l} x \mathcal{L}_H y \Rightarrow y^{-1}x \in H \\ y \mathcal{L}_H z \Rightarrow z^{-1}y \in H \end{array}\right] \overset{H \leq G}{\Longrightarrow} H \ni (z^{-1}y)(y^{-1}x) = z^{-1}\underbrace{(yy^{-1})}_{= e}x = z^{-1}x$$

NEED TO PROVE $x \mathcal{L}_H z$, THAT IS $z^{-1}x \in H$ ✓

2. Find, if possible, an element in S_5 with the indicated order. Justify your answer or choice with explicit computations or arguments based on results from our book.

   (a) Order 6.

   (b) Order 13.

(a) $\sigma = (12)(345)$ HAS ORDER 6, BECAUSE:

$(12)^2 = (1)$; $(345)^2 = (354)$, $(345)^3 = (1)$;

THEN $|\sigma|$ MUST BE DIVISIBLE BY BOTH 2 AND 3, AND

$\sigma^6 = (12)^6 (345)^6 = ((12)^2)^3 \cdot ((345)^3)^2 = (1)$.

(b) BY LAGRANGE THEOREM $|x| = |\langle x \rangle|$ DIVIDES $|G|$ FOR ALL $x \in G$, BUT HERE $G = S_5$ AND $|S_5| = 5! = 120$ IS NOT DIVISIBLE BY 13. THEREFORE, THERE CANNOT BE $x \in S_5$ WITH $|x| = 13$.

3. Let G be a (multiplicative) group and $x, y \in G$. Define the **commutator** of x and y to be the element $[x, y] = xyx^{-1}y^{-1}$, then define the **first derived subgroup** of G to be $G' = \langle [x, y] \in G \mid x, y \in G \rangle$ (the set of finite products of elements $[x, y]$). Prove that $G' \leq G$ (that is G' is a subgroup of G).

INDEED, $e = [e, e] \in G'$, THEN $G' \neq \emptyset$ AND $e \in G'$

BY CONSTRUCTION G' IS CLOSED UNDER PRODUCT, THEREFORE WE ONLY NEED TO PROVE THAT EACH ELEMENT HAS INVERSE IN G':

$a \in G'$ AND $a = [x_1, y_1] \cdots [x_n, y_n]$

PROVE THAT $a^{-1} = [y_n, x_n] \cdots [y_1, x_1] \in G'$ FOR $n \in \mathbb{N}$

$\boxed{\text{BASE}}$ $n = 1$: $a = [x, y] \Rightarrow a^{-1} = (xyx^{-1}y^{-1})^{-1} =$

$= (y^{-1})^{-1}(x^{-1})^{-1}y^{-1}x^{-1} = yxy^{-1}x^{-1} = [y, x] \in G'$ ✓

$\boxed{\text{STEP}}$ ASSUME $\left([x_1, y_1] \ldots [x_n, y_n]\right)^{-1} = [y_n, x_n] \ldots [y_1, x_1]$ (H.I.)

PROVE THAT $\left([x_1, y_1] \ldots [x_{n+1}, y_{n+1}]\right)^{-1} = [y_{n+1}, x_{n+1}] \ldots [y_1, x_1]$

$$\left([x_1, y_1] \ldots [x_n, y_n] \cdot [x_{n+1}, y_{n+1}]\right)^{-1} =$$

$$= \underbrace{[x_{n+1}, y_{n+1}]^{-1}}_{\text{AS FOR BASE}} \cdot \underbrace{\left([x_1, y_1] \ldots [x_n, y_n]\right)^{-1}}_{\text{H.I.}} =$$

$$= [y_{n+1}, x_{n+1}] \cdot \left([y_n, x_n] \cdots [y_1, x_1]\right)$$ ✓

NOTE THAT WE PROVED THAT: $\left([x_1, y_1] \ldots [x_n, y_n]\right)^{-1} = [y_n, x_n] \ldots [y_1, x_1]$

EXAMPLE:

IN S_4 $[(12), (23)] \cdot [(13), (34)] = (12)(23)(12)^{-1}(23)^{-1}(13)(34)(13)^{-1}(34)^{-1}$

$= (12)(23)(12)(23)(13)(34)(13)(34) = (142)$, THEN

$\left([(12), (23)][(13), (34)]\right)^{-1} = (142)^{-1} = (124)$

USING THE ABOVE PROOF: $[y_2, x_2] \cdot [y_1, x_1] =$

$= [(34), (13)] \cdot [(23), (12)] = (34)(13)(34)(13)(23)(12)(23)(12) = (124)$

MOREOVER, THESE PRODUCTS, (142) AND (124), NEED NOT TO BE COMMUTATORS!

4. Let $H = \langle (1324) \rangle$ be a cyclic subgroup of S_4. Compute the quotient set $\dfrac{S_4}{\mathcal{R}_H}$, that is the partition of S_4 generated by the right-cosets of H.

$$\left.\begin{aligned} &H = \langle (1324)^n \in S_4 \mid n \in \mathbb{N} \rangle \\ &(1324)^2 = (1324)(1324) = (12)(34) \\ &(1324)^3 = (12)(34)(1324) = (1423) \\ &(1324)^4 = (1423)(1324) = (1) \end{aligned}\right\} \Rightarrow H = \{(1), (1324), (12)(34), (1423)\}$$

$|H| = 4$ AND $|S_4 : H| = \frac{24}{4} = 6$

SO THAT $S_4/\mathcal{R}_H$ HAS 6 DISTINCT COSETS.

First let's list the 24 elements of S_4: (THEN APPLY 6.3 FOR RIGHT COSETS)

$(1), (1\,2), (1\,3), (1\,4), (2\,3), (2\,4), (3\,4)$	these fix more than one element,
$(1\,2\,3), (1\,3\,2), (1\,2\,4), (1\,4\,2), (1\,3\,4), (1\,4\,3), (2\,3\,4), (2\,4\,3)$	these fix only one element,
$(1\,2\,3\,4), (1\,2\,4\,3), (1\,3\,2\,4), (1\,3\,4\,2), (1\,4\,3\,2), (1\,4\,2\,3)$	cycles which do not fix any element,
$(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)$	non-cycles which do not fix any element.

$$H = H(1) = H(1324) = H(12)(34) = H(1423)$$

$$H(12) = \{(12), (1324)(12), (12)(34)(12), (1423)(12)\}$$
$$= \{(12), (14)(23), (34), (13)(24)\} = H(14)(23) = H(34) = H(13)(24)$$

$$H(13) = \{(13), (1324)(13), (12)(34)(13), (1423)(13)\}$$
$$= \{(13), (124), (1432), (234)\} = H(124) = H(1432) = H(234)$$

$$H(14) = \{(14), (1324)(14), (12)(34)(14), (1423)(14)\}$$
$$= \{(14), (243), (1342), (123)\} = H(243) = H(1342) = H(123)$$

$$H(23) = \{(23), (1324)(23), (12)(34)(23), (1423)(23)\}$$
$$= \{(23), (134), (1243), (142)\} = H(134) = H(1243) = H(142)$$

LAST ONE HAS NO CHOICES:

$$H(24) = \{(24), (1324)(24), (12)(34)(24), (1423)(24)\}$$
$$= \{(24), (132), (1234), (143)\} = H(132) = H(1234) = H(143)$$

THEN $\dfrac{S_4}{\mathcal{R}_H} = \{H, H(12), H(13), H(14), H(23), H(24)\}$

5. List the left and right cosets of $\langle \bar{9} \rangle$ in $\mathbb{Z}_{10}^* = U(10)$.

$$\mathbb{Z}_{10}^* = U(10) = \{ \bar{1}, \bar{3}, \bar{7}, \bar{9} \}$$

$$\left.\begin{array}{l} H = \langle \bar{9} \rangle = \{ \bar{9}^n \mid n \in \mathbb{N} \} \\ \bar{9}^2 = \overline{81} = \bar{1} \end{array}\right] \Rightarrow \langle \bar{9} \rangle = \{ \bar{1}, \bar{9} \} = H$$

BECAUSE $(\mathbb{Z}_{10}^*, \cdot)$ IS ABELIAN, THAT IS $\cdot$ IS COMMUTATIVE, LEFT AND RIGHT COSETS ARE THE SAME: $gH = Hg$.

$$H = \bar{1}H = \bar{9}H$$

$$\bar{3}H = \{ \bar{3}, \bar{3} \cdot \bar{9} \} = \{ \bar{3}, \bar{7} \} = \bar{7}H$$

$$\mathbb{Z}_{10}^* / \langle \bar{9} \rangle = \{ \langle \bar{9} \rangle, \bar{3} \langle \bar{9} \rangle \}$$

Only for Math 310-02H

6. Find the subgroup of $GL_2(\mathbb{R})$ generated by $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

$$\left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^n \in GL_2(\mathbb{R}) \mid n \in \mathbb{Z} \right\}$$

WE CAN USE A MACHINE TO COMPUTE THESE POWERS

A:={{1,-1},{1,0}}

$\rightarrow A := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

A^2

$\rightarrow \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

A^3

$\rightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

A^4

$\rightarrow \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

A^5

$\rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

A^6

$\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

THEREFORE $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ HAS ORDER 6

NOTE THAT: $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^{-1} = A^5$; $A^2 = -A^{-1}$

$A^4 = -A$; $A^3 = -I$

THEN:

$$\langle A \rangle = \{ I, -I, A, -A, A^{-1}, -A^{-1} \}$$

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1. List the elements of S_4 and of its cyclic subgroup $\langle(134)\rangle$, then compute all the left cosets of $\langle(134)\rangle$ in S_4.

LIKE #4 TAKE-HOME.

$(134)^2 = (134)(134) = (143) \Rightarrow (134)^3 = (143)(134) = (1) \Rightarrow$

$\Rightarrow H = \langle(134)\rangle = \{(1), (134), (143)\}$. BY LAGRANGE, $|S_4 : H| = \frac{24}{3} = 8$

THEN $S_4/\mathcal{L}_H$ HAS 8 DISTINCT COSETS.

First let's list the 24 elements of S_4: (THEN APPLY 6.3 FOR RIGHT COSETS)

$(1), (1\,2), (1\,3), (1\,4), (2\,3), (2\,4), (3\,4)$	these fix more than one element,
$(1\,2\,3), (1\,3\,2), (1\,2\,4), (1\,4\,2), (1\,3\,4), (1\,4\,3), (2\,3\,4), (2\,4\,3)$	these fix only one element,
$(1\,2\,3\,4), (1\,2\,4\,3), (1\,3\,2\,4), (1\,3\,4\,2), (1\,4\,3\,2), (1\,4\,2\,3)$	cycles which do not fix any element,
$(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)$	non-cycles which do not fix any element.

$H = (1)H = (134)H = (143)H$

$(12)H = \{(12), (12)(134), (12)(143)\} = \{(12), (1342), (1432)\}$.
$= (1342)H = (1432)H$

$(13)H = \{(13), (13)(134), (13)(143)\} = \{(13), (34), (14)\} = (34)H = (14)H$

$(23)H = \{(23), (23)(134), (23)(143)\} = \{(23), (1234), (1423)\}$
$= (1234)H = (1423)H$

$(24)H = \{(24), (24)(134), (24)(143)\} = \{(24), (1324), (1243)\}$
$= (1324)H = (1243)H$

$(123)H = \{(123), (123)(134), (123)(143)\} = \{(123), (234), (14)(23)\}$
$= (234)H = (14)(23)H$

$(132)H = \{(132), (132)(134), (132)(143)\} = \{(132), (12)(34), (142)\}$
$= (12)(34)H = (142)H$

THE LAST ONE IS DETERMINED:

$(124)H = \{(124), (124)(134), (124)(143)\} = \{(124), (13)(24), (243)\}$
$= (13)(24)H = (243)H$.

THEREFORE

$S_4/\mathcal{L}_H = \{H, (12)H, (13)H, (23)H, (24)H, (14)(23)H, (12)(34)H, (13)(24)H\}$

2. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & 6 & 4 & 5 & 7 & 9 & 8 \end{pmatrix},$$

(a) Write a disjoint-cycles-decomposition of σ.

(b) Compute the order of σ.

(a) $\sigma = (1\,3\,2)(4\,6\,5)(8\,9)$

(b) $|(1\,3\,2)| = |(4\,6\,5)| = 3$ AND $|(8\,9)| = 2 \Rightarrow$

$\Rightarrow$ 2 AND 3 DIVIDE $|\sigma|$.

$\sigma^6 = (1\,3\,2)^6 (4\,6\,5)^6 (8\,9)^6 = \left((1\,3\,2)^3\right)^2 \left((4\,6\,5)^3\right)^2 \cdot \left((8\,9)^2\right)^3 = (1)$

THEN $|\sigma| = 6$