Mat321 – Spring 2019 – Test4

Instructor: Dr. Francesco Strazzullo **Name** KEY

I certify that I did not receive third party help in *completing* this test (sign) ____________

Instructions. Technology and instructor's notes (including the formula sheets from our book) are allowed on this exam. Each problem is worth 10 points. If you use notes or formula sheets, make a reference. When using technology describe which commands (or keys typed) you used or print out your worksheet.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Consider the recursive sequence defined by

$$x_1 = 3\,; \;\; x_{n+1} = 2 - \frac{1}{x_n}\,, \qquad n > 1.$$

a) Evaluate the first three terms of this sequence.
b) You can assume the sequence to be monotonic and bounded (and hence convergent). Indicate if it is increasing or decreasing, finding a bounding interval.
c) Find the limit of this sequence.

a) $x_1 = 3$; $x_2 = 2 - \frac{1}{3} = \frac{5}{3}$; $x_3 = 2 - \frac{3}{5} = \frac{7}{5}$; ($x_4 = 2 - \frac{3}{7} = \frac{11}{7}$; $x_5 = 2 - \frac{7}{11} = \frac{16}{11}$; ...)

b) SEQUENCE IS DECREASING WITH $0 \leq x_n \leq 3$

c) IF L IS THE LIMIT, THEN $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = L$

AND THE RECURSION FORMULA GIVES $L = 2 - \frac{1}{L}$. THEN

L IS A SOLUTION OF $x = 2 - \frac{1}{x}$ WITH $0 \leq L \leq 3$.

$x = 2 - \frac{1}{x} \Rightarrow x^2 = 2x - 1 \Rightarrow x^2 - 2x + 1 = 0 \Rightarrow$

$\Rightarrow (x-1)^2 = 0 \Rightarrow x = 1 \Rightarrow L = 1$

2. Prove divergent or find the sum of the series

$$\sum_{n=4}^{\infty} \frac{3}{n(n-3)} = 3\sum_{n=4}^{\infty} \frac{1}{n(n-3)} = 3\sum_{i=4}^{\infty} \frac{-1/3}{i} + \frac{1/3}{i-3} = \sum_{i=4}^{\infty} \frac{1}{i-3} - \frac{1}{i}$$

$$S_n = \sum_{i=4}^{n} \frac{1}{i-3} - \frac{1}{i} = \frac{1}{1} - \cancel{\frac{1}{4}} + \frac{1}{2} - \cancel{\frac{1}{5}} + \frac{1}{3} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{4}} - \frac{1}{7} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{8}} + \dots$$

$$+ \cancel{\frac{1}{n-6}} - \cancel{\frac{1}{n-3}} + \cancel{\frac{1}{n-5}} - \frac{1}{n-2} + \cancel{\frac{1}{n-4}} - \frac{1}{n-1} + \cancel{\frac{1}{n-3}} - \frac{1}{n} =$$

$$= 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n-2} - \frac{1}{n-1} - \frac{1}{n} \xrightarrow[n\to\infty]{} \frac{11}{6} + 0 = \frac{11}{6} \Rightarrow$$

$$\Rightarrow \sum_{n=4}^{\infty} \frac{3}{n(n-3)} = \lim_{n\to\infty} S_n = \frac{11}{6}$$

3. Prove divergent or find the sum of the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n(n+4)}$$

ALL POSITIVE

RATIO TEST $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)(n+5)} \cdot \frac{n(n+4)}{2^n} = 2\underbrace{\frac{n(n+4)}{(n+1)(n+5)}}_{\to 1} \to 2$

POLYNOMIALS OF DEG. 2 AT NUM. AND DENOM.

$2 > 1 \Rightarrow$ DIVERGENT

—

DIV. TEST: $f(x) = \frac{2^x}{x(x+4)} \xrightarrow[x\to\infty]{HR} \frac{2^x}{2x+4} \xrightarrow[x\to\infty]{HR} \frac{2^x}{2} = 2^{x-1} \xrightarrow[x\to\infty]{} \infty$

4. Find the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n!\,3^{n-1}}$$

RATIO TEST: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)^{n+1}}{(n+1)!\,3^n} \cdot \frac{n!\,3^{n-1}}{(x-2)^n}\right| =$

$= \left|\frac{(x-2)}{3(n+1)}\right| \xrightarrow[n\to\infty]{} 0$ $\Rightarrow$ ALWAYS CONVERGENT $\Rightarrow$

$\Rightarrow R = \infty$

5. Find the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n!}$$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{(n+1)!} \frac{n}{(x-3)^n}\right| = \left|\frac{(x-3)}{n+1}\right| \xrightarrow[n\to\infty]{} 0$

$R = \infty.$

6. Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(2x-5)^n}{3n^3}$$

$$\left| a_{n+1} \cdot \frac{1}{a_n} \right| = \left| \frac{(2x-5)^{n+1}}{3(n+1)^3} \cdot \frac{3n^3}{(2x-5)^n} \right| =$$

$$= |2x-5| \cdot \left(\frac{n}{n+1}\right)^3 \xrightarrow[n \to \infty]{} |2x-5| \cdot 1^3 < 1 \Rightarrow |2x-5| < 1 \Rightarrow$$

$$\Rightarrow -1 < 2x-5 < 1 \Rightarrow 2 < x < 3.$$

CHECK ENDPOINTS: • $x = 2 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{3n^3}$ ALTERNATING WITH $b_n = \frac{1}{3n^3} \to 0$ AND DECREASING $\Rightarrow$ SERIES CONVERGENT.

• $x = 3 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{3n^3}$ WITH $\int_1^{\infty} \frac{1}{3x^3}\,dx = \frac{1}{3} \lim_{t \to \infty} \int_1^t x^{-3}\,dx = \frac{1}{3}\left[\frac{x^{-2}}{-2}\right]_1^{\infty} = -\frac{1}{6}(0-1) \Rightarrow$

$\Rightarrow$ CONV. BY INTEGR. TEST.

INT. OF CONV $= [2, 3]$

7. Find the coefficient of x^3 in the Maclaurin series for $f(x) = x^4 - \sin(2x)$.

$$f^{(1)}(x) = 4x^3 - 2\cos(2x) \Rightarrow f^{(2)}(x) = 12x^2 + 4\sin(2x)$$

$$M_f = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i \Rightarrow \text{COEFF OF } x^3 = \frac{f^{(3)}(0)}{3!}$$

$$\Rightarrow f^{(3)}(x) = 24x + 8\cos(2x) \quad \text{WITH} \quad \frac{f^{(3)}(0)}{3!} = \frac{8}{6} = \frac{4}{3}.$$

Mat321 – Spring 2019 – Test4 – In Class

Instructor: Dr. Francesco Strazzullo **Name**______________________________

Instructions. Technology and instructor's notes (including the formula sheets from our book) are allowed on this exam. Each problem is worth 10 points. If you use notes or formula sheets, make a reference. When using technology describe which commands (or keys typed) you used or print out your worksheet.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

8. Find the radius and the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(3x+4)^n}{2^{n+1}}$$

$$\left| a_{n+1} \cdot \frac{1}{a_n} \right| = \left| \frac{(3x+4)^{n+1}}{2^{n+2}} \cdot \frac{2^{n+1}}{(3x+4)^n} \right| = \left| \frac{1}{2}(3x+4) \right| < 1 \Rightarrow$$

$$\Rightarrow -2 < 3x+4 < 2 \Rightarrow -2 < x < -2/3 \Rightarrow R = \frac{1}{2}\left(-\frac{2}{3}+2\right) = \frac{2}{3}$$

ENDPOINTS: • $x = -2 \Rightarrow \sum_{n=0}^{\infty} \frac{(-2)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2}(-1)^n$ DIVERGENT BY DIV. TEST.

• $x = \frac{2}{3} \Rightarrow \sum_{n=0}^{\infty} \frac{6^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2} 3^n$ DIVERGENT BY DIV. TEST.

INT. CONV $= \left(-2, -\frac{2}{3}\right)$

9. Prove divergent or find the sum of the series

$$\sum_{n=5}^{\infty} \frac{2}{n^2-16} = 2 \sum_{n=5}^{\infty} \frac{1}{(n-4)(n+4)} = 2 \sum_{i=5}^{\infty} \frac{1/8}{i-4} - \frac{1/8}{i+4} = \frac{1}{4} \sum_{i=5}^{\infty} \frac{1}{i-4} - \frac{1}{i+4}$$

$$4S_n = \sum_{i=5}^{n} \frac{1}{i-4} - \frac{1}{i+4} = 1 - \cancel{\frac{1}{9}} + \frac{1}{2} - \cancel{\frac{1}{10}} + \frac{1}{3} - \cancel{\frac{1}{11}} + \frac{1}{4} - \cancel{\frac{1}{12}} + \ldots + \cancel{\frac{1}{9}} - \cancel{\frac{1}{17}} +$$

$$+ \cancel{\frac{1}{10}} - \cancel{\frac{1}{18}} + \ldots + \frac{1}{n-9} - \cancel{\frac{1}{n-1}} + \frac{1}{n-8} - \frac{1}{n} + \frac{1}{n-7} - \frac{1}{n+1} + \frac{1}{n-6} - \frac{1}{n+2}$$

$$+ \frac{1}{n-5} - \frac{1}{n+3} + \frac{1}{n-4} - \frac{1}{n+4} \xrightarrow[n\to\infty]{} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + 0 =$$

$$= \frac{761}{280} \Rightarrow \sum_{n=5}^{\infty} \frac{2}{n^2-16} = \frac{1}{4} \cdot \frac{761}{280} = \frac{761}{1120}$$

10. Express the 3rd-degree Taylor polynomial centered at $x = 1$ for $f(x) = x^3 - \frac{2}{\pi}\cos(\pi x)$. Write it as a sum of powers of binomials, do not expand. (HONOR: write the 4th-degree one)

$f'(x) = 3x^2 + 2\sin(\pi x) \Rightarrow f'(1) = 3$

$f''(x) = 6x + 2\pi\cos(\pi x) \Rightarrow f''(1) = 6 - 2\pi$

$f^{(3)}(x) = 6 - 2\pi^2\sin(\pi x) \Rightarrow f^{(3)}(1) = 6$

$$T^n_{f,c} = \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(x-c)^i$$

$$T^3_{f,1} = \sum_{i=0}^{3} \frac{f^{(i)}(1)}{i!}(x-1)^i = 1 + \frac{2}{\pi} + 3(x-1) + (3-\pi)(x-1)^2 + (x-1)^3$$

Honor $+ \frac{\pi^3}{12}(x-1)^4$

$f^{(4)}(x) = -2\pi^3\cos(\pi x) \Rightarrow f^{(4)}(1) = 2\pi^3$