Math 221- Fall 2012 - Final - Part 1

Instructor: Dr. Francesco Strazzullo My Name KEY

I certify that I did not receive third party help in *completing* Part 1 of this test. (*sign*)

Instructions. You **can not** use a graph to justify your answer. Each problem is worth 10 points.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Evaluate the integral $\int_0^5 2e^x + 4\cos x\, dx$.

$\int e^x dx = e^x + c$ AND $\int \cos x\, dx = \sin x\, dx + c$

$$\int_0^5 2e^x + 4\cos x\, dx = \left[2e^x + 4\sin x\right]_0^5 = 2e^5 + 4\sin 5 - (2e^0 + 4\sin 0)$$

$$\approx 290.99$$

2. Water flows from the bottom of a storage tank at a rate of $V'(t) = 200 - 4t$ liters per minute, where $0 \le t \le 50$. Find the amount of water that flows from the thank during the first 10 minutes.

"AMOUNT OF WATER FLOWING" = "NET CHANGE DURING FIRST 10 minutes" =

$$= V(10) - V(0) = \int_0^{10} V'\, dt = \int_0^{10} 200 - 4t\, dt =$$

$$= \left[200t - 4\frac{t^2}{2}\right]_0^{10} = 2\left[100t - t^2\right]_0^{10} = 2(1000 - 100) = 1800 \text{ LITERS.}$$

3. The error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

is used in probability, statistics, and engineering. Show that the function

$$y = e^{x^2} \operatorname{erf}(x)$$

satisfies the first order ODE

$$y' = 2xy + \frac{2}{\sqrt{\pi}}.$$

$$\operatorname{erf}(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-t^2}\,dt \overset{\text{By}}{\underset{\text{FTC}}{\Longrightarrow}} \operatorname{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

ODE:

$$\text{L.H.S} = y' = \frac{d}{dx}\left[e^{x^2}\right] \cdot \operatorname{erf}(x) + e^{x^2} \cdot \frac{d}{dx}\left[\operatorname{erf}(x)\right]$$

$$= (2x)\,\underbrace{e^{x^2}\operatorname{erf}(x)}_{y} + e^{x^2}\operatorname{erf}'(x)$$

$$= 2x\,y + \cancel{e^{x^2}} \cdot \frac{2}{\sqrt{\pi}} \cancel{e^{-x^2}}$$

$$= 2xy + \frac{2}{\sqrt{\pi}} = \text{R.H.S.} \quad \checkmark$$

4. Nine elks are introduced into a game preserve. It is estimated that their population will increase yearly with rate

$$p'(t) = \frac{54e^{-t}}{(1+2e^{-t})^2}.$$

Using the substitution method find the net increase in population between the first and the third year.

"NET INCREASE FIRST AND 3rd YEAR" $= p(3) - p(1) = \int_1^3 p'(t)\,dt$

$$= \int_1^3 \frac{54e^{-t}}{(1+2e^{-t})^2}\,dt = 54\int_{1+2e^{-1}}^{1+2e^{-3}} \frac{e^{-t}}{u^2}\,\frac{du}{-2e^{-t}} =$$

SUBSTITUTION : $u = 1+2e^{-t}$

$du = -2e^{-t}\,dt \rightarrow dt = -\frac{1}{2e^{-t}}\,du$

NEW ENDPOINTS $\left[\begin{array}{l} t=1 \rightarrow u = 1+2e^{-1} \\ t=3 \rightarrow u = 1+2e^{-3} \end{array}\right.$

$$= -27\int_{1+2e^{-1}}^{1+2e^{-3}} u^{-2}\,du = -27\left[\frac{u^{-2+1}}{-2+1}\right]_{1+2e^{-1}}^{1+2e^{-3}} =$$

$$= 27\left[\frac{1}{u}\right]_{1+2e^{-1}}^{1+2e^{-3}} = 27\left(\frac{1}{1+2e^{-3}} - \frac{1}{1+2e^{-1}}\right)$$

$\approx 8.9998 \rightarrow$ ABOUT 9 ELKS INCREASE.

5. Use integration by parts to compute the most general antiderivative $F(x)$ of $f(x) = x\sin(2x)$, then check your answer by differentiation.

a) $\int x \sin(2x)\,dx$

$u = x \to du = dx$

$dv = \sin(2x)dx \to v = \int dv = \int \sin(2x)\,dx = \frac{1}{2}\int \sin t\,dt = \frac{1}{2}(-\cos(2x))$

$t = 2x$

$\int u\,dv = uv - \int v\,du$

$\int x\sin(2x)\,dx = x\left(-\frac{1}{2}\cos(2x)\right) - \int -\frac{1}{2}\cos(2x)\,dx$

AS ABOVE

$= -\frac{1}{2}x\cos(2x) + \frac{1}{2}\int \cos(2x)\,dx$

$= -\frac{1}{2}x\cos(2x) + \frac{1}{2}\left(\frac{1}{2}\sin(2x)\right) + C \Rightarrow$

$\Rightarrow \boxed{F(x) = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C}$

b) $F'(x) = -\frac{1}{2}\left(1\cdot\cos(2x) + x\left((2)(-\sin(2x))\right)\right) + \frac{1}{4}(2)\cos(2x) + 0$

$= -\frac{1}{2}\cos(2x) + x\sin(2x) + \frac{1}{2}\cos(2x)$ ✓

Math 221- Fall 2012 - Final - Part 2/2

Instructor: Dr. Francesco Strazzullo Name KEY

Instructions. You **can not** use a graph to justify your answer. Each problem is worth 15 points.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

Compute the following integrals.

1. $\int \frac{1}{x\sqrt[3]{\ln x}}\, dx$

SUBS: $u = \ln x$

$\Downarrow$

$du = \frac{1}{x} dx \Rightarrow dx = x\, du$

$$\rightarrow \int \frac{1}{x(\ln x)^{\frac{1}{3}}} dx = \int \frac{(\ln x)^{-\frac{1}{3}}}{x} dx = \int \frac{u^{-\frac{1}{3}}}{x} x\, du$$

$$= \int u^{-\frac{1}{3}} du = \frac{u^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C = \frac{u^{2/3}}{2/3} + C$$

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

$$= \frac{3}{2}(\ln x)^{2/3} + C$$

2. $\int_6^8 \frac{4}{(6-x)^3}\,dx$ THE INTEGRAND $\frac{4}{(6-x)^3}$ IS UNDEFINED AT $X=6$!

$$\int_6^8 \frac{4}{(6-x)^3}\,dx = \lim_{t\to 6^+} \int_t^8 \frac{4}{(6-x)^3}\,dx$$

$$= \lim_{t\to 6^+} 4\int_t^8 (6-x)^{-3}\,dx =$$

SUBST.: $u = 6-x \Rightarrow du = -dx \Rightarrow dx = -du$

NEW ENDPOINTS: $x=t \to u=6-t$; $x=8 \to u=-2$

$$= 4\lim_{t\to 6^+} \int_{6-t}^{-2} u^{-3}(-du) = 4\lim_{t\to 6^+} \int_{-2}^{6-t} u^{-3}\,du$$

$$= 4\lim_{t\to 6^+} \left[\frac{u^{-3+1}}{-3+1}\right]_{-2}^{6-t} = 4\lim_{t\to 6^+} \left[-\frac{1}{2}u^{-2}\right]_{-2}^{6-t}$$

$$= -2\lim_{t\to 6^+} \left[\frac{1}{(6-t)^2} - \frac{1}{4}\right] \approx -2\left(\frac{1}{0^+} - \frac{1}{4}\right)$$

$$\approx -2\left(+\infty - \frac{1}{4}\right) = -\infty \quad \text{DIVERGENT}$$

3. $\int_1^4 \frac{\ln y}{\sqrt{y}}\, dy$

(THIS SHOULD BE BY PARTS)

TRY SUBS: $u = \ln y \Rightarrow du = \frac{1}{y} dy \Rightarrow dy = y\, du \Rightarrow$

$\int \frac{\ln y}{y^{\frac{1}{2}}} dy = \int u \cdot y^{-\frac{1}{2}} \cdot y\, du = \int u\, y^{\frac{1}{2}}\, du$; Y DOESN'T DIVIDE OUT!

$$\int u\, dv = uv - \int v\, du$$

$$\int \underbrace{y^{-\frac{1}{2}} \overbrace{\ln y}^{u}\, dy}_{dv} = (\ln y)\, 2y^{\frac{1}{2}} - \int 2y^{\frac{1}{2}} \frac{1}{y}\, dy =$$

$u = \ln y \Rightarrow du = \frac{1}{y} dy$

$dv = y^{-\frac{1}{2}} dy \Rightarrow v = \int dv = \int y^{-\frac{1}{2}} dy = \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2y^{\frac{1}{2}}$

$$= 2y^{\frac{1}{2}} \ln y - 2\int y^{-\frac{1}{2}} dy = 2\sqrt{y}\, \ln y - 2\left(2\sqrt{y}\right) + C$$

$$= 2\sqrt{y}\left(\ln y - 2\right) + C$$

THUS:

$$\int_1^4 \frac{\ln y}{\sqrt{y}}\, dy = \left[2\sqrt{y}\left(\ln y - 2\right)\right]_1^4 = 2\left(\sqrt{4}\left(\ln 4 - 2\right) - \sqrt{1}\left(\ln 1 - 2\right)\right)$$

$\left[= 2\left(2\ln 4 - 4 + 2\right) = 4\left(\ln 4 - 1\right)\right]$ NOT NEEDED

≈ 1.54518