Notes

Let $(G, \cdot)$ be a *finite* group with *multiplicative notation*, then

- for $a \in G$ and $b \in G$ we usually write $a \cdot b$ as ab;
- u denotes the *neutral element of* G;
- the inverse of $a \in G$ is denoted by a^{-1};
- $|G|$ denotes the *order of* G, that is the number of elements in G. For instance S_n, the symmetric group on n elements, is such that $|S_n| = n! = n \cdot (n-1) \ldots 3 \cdot 2$;
- for every $x \in G$ the powers of x are defined by *recursion* (induction) as
  1. $x^0 = u$,
  2. $x^n = x^{n-1}x$, for every $n \in \mathbb{N}$,

  and the least natural number n such that $x^n = u$ is called the order of x (it is defined because $\mathbb{N}$ is well-ordered and G is finite) and the order of x is denoted by $\circ(x)$ or $|x|$; in additive notation we define the n-th multiples of x by
  1. $0x = u$,
  2. $nx = n - 1x + x$, for every $n \in \mathbb{N}$.
- if $(F, \odot)$ is another group and $f : G \mapsto F$ is a function, we say that f is a *homomorphism* of groups when for all $a \in G$ and $b \in G$ it is
$$f(a \cdot b) = f(a) \odot f(b).$$
  Moreover, we say that f is an *isomorphism* when it is also bijective, that is 1-to-1 and onto, and we say that G and F are isomorphic, denoting $G \cong F$.

By definition a subset $H \subset G$ is a subgroup of G if $(H, \cdot)$ is a group: in particular $u \in H$ and H can not be empty. For instance, if $f : G \mapsto F$ is a homomorphism of groups then $\ker f = \{a \in G \text{ such that } f(a) = u_F\}$ is a subgroup of G (note, u_F is the neutral element in F).

Proposition 0.1 *H is a subgroup of G if and only if*

$$a, b \in H \Rightarrow ab^{-1} \in H.$$

We use the notation $H \leq G$ when H is a subgroup of G.

Proposition 0.2 *Let $x \in G$, then*
$$\langle x \rangle = \{x^n \in G : n \in \mathbb{N}\}$$
is a subgroup of G, called cyclic *subgroup generated by x. G is said cyclic if $G = \langle x \rangle$, in which case x is called a generator of G.*

Theorem 0.3 (Lagrange) *The order of a subgroup divides the order of the group, that is*

$$H \leq G \Rightarrow |G| = r|H|,$$

by definition $r = |G : H|$ is called the index of H in G.
In particular, the order of every element of G divides the order of G.

Theorem 0.4 (Cayley) *If $|G| = n$ then G is the isomorphic copy of a subgroup of S_n, the symmetric group on n elements.*

Proposition 0.5 *Let $H \leq G$, $a \in G$, and $b \in G$, then*

1. *the relation*
$$a\,\mathcal{L}_H\,b \Leftrightarrow b^{-1}a \in H$$
defines an equivalence relation on G*, whose classes of equivalence are the left cosets*
$$aH = \{ah \in G, \textit{for all } h \in H\}\,.$$

2. *the relation*
$$a\,\mathcal{R}_H\,b \Leftrightarrow ab^{-1} \in H$$
defines an equivalence relation on G*, whose classes of equivalence are the right cosets*
$$Ha = \{ha \in G, \textit{for all } h \in H\}\,.$$

Theorem 0.6 *Consider the equivalence relations of Proposition 0.5, then the following conditions are equivalent to each other:*

1. *The equivalence relations are the same, that is*
$$a\,\mathcal{R}_H\,b \Leftrightarrow a\,\mathcal{L}_H\,b;$$

2. *for all* $g \in G$ *it is* $gH = Hg$*;*

3. *for all* $g \in G$ *and* $h \in H$ *it is* $ghg^{-1} \in H$*;*

4. *for all* $g \in G$ *it is* $H = gHg^{-1} = \{ghg^{-1} \in G \mid h \in H\}$*;*

5. $(G/\mathcal{L}_H, \odot)$ *and* $(G/\mathcal{R}_H, \odot)$ *are isomorphic groups, where the multiplications are defined by*
$$aH \odot bH = (ab)H$$
and
$$Ha \odot Hb = H(ab)$$

If any of the conditions of Theorem 0.6 is satisfied, then H is said to be a *normal* (or invariant) subgroup of G and we write $H \lhd G$. Moreover, we denote by $(G/H, \cdot)$ any of the groups $(G/\mathcal{L}_H, \odot)$ or $(G/\mathcal{R}_H, \odot)$, and we call it *the quotient group of* G *by* H.
Look at examples 8, 9, 10, and 13 of chapter 9 and at exercises 9.26, 9.27, and 9.36, from the textbook "*Abstract Algebra*", by Lloyd Jaisingh and Frank Ayres, Schaum's Outlines (Mc Graw Hill), IBN10: 0071403272. Notes from our current textbook are on EagleWeb.

Math 310-010 - Spring 2014 - Test 3 - Solutions

Instructor: Dr. Francesco Strazzullo
Instructions. SHOW YOUR WORK neatly, please. Each exercise is worth 10 points. If using a result from our textbook, make a reference to it (using the page number as well). You might also use results included in the notes attached to this test.

1. Show that the multiplicative group $\left(\mathbb{Z}_7^\times, \cdot\right)$ is isomorphic to the additive group $(\mathbb{Z}_6, +)$ by at least
   (a) providing operation tables for both groups, and
   (b) describing a mapping $f : \mathbb{Z}_6 \mapsto \mathbb{Z}_7$.

Solution When possible, the notation $\bar{a} = [a]_m$ is used.

(a) Operation tables: the operations are both commutative, therefore we don't need to write the lower triangular part of the tables.

$(\mathbb{Z}_7^\times, \cdot)$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
$\bar{2}$		$\bar{4}$	$\bar{6}$	$\bar{1}$	$\bar{3}$	$\bar{5}$
$\bar{3}$			$\bar{2}$	$\bar{5}$	$\bar{1}$	$\bar{4}$
$\bar{4}$				$\bar{2}$	$\bar{6}$	$\bar{3}$
$\bar{5}$					$\bar{4}$	$\bar{2}$
$\bar{6}$						$\bar{1}$

Note: $\bar{2}^2 = \bar{4}$, $\bar{2}^3 = 1$, then $\circ(\bar{2}) = 3$; $\bar{3}^2 = \bar{2}$, $\bar{3}^6 = \left(\bar{3}^2\right)^3 = \bar{2}^3 = \bar{1}$, then $\circ(\bar{3}) = 6$ and $\bar{3}$ is a generator of $\mathbb{Z}_7^\times$. Therefore $\mathbb{Z}_7^\times$ is cyclic with $\mathbb{Z}_7^\times = \langle [3]_7 \rangle = \left\{ [3]_7, {[3]_7}^2 = \bar{2}, \ldots, {[3]_7}^6 = \bar{1} \right\}$.

$(\mathbb{Z}_6, +)$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{1}$		$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{0}$
$\bar{2}$			$\bar{4}$	$\bar{5}$	$\bar{0}$	$\bar{1}$
$\bar{3}$				$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{4}$					$\bar{2}$	$\bar{3}$
$\bar{5}$						$\bar{4}$

Note: $(\mathbb{Z}_6, +)$ is the standard cyclic group of order 6, with generator $[1]_6$, because $\mathbb{Z}_6 = \langle [1]_6 \rangle = \{[1]_6, 2[1]_6 = \bar{2}, \ldots, 6[1]_6 = \bar{0}\}$.

(b) To describe an isomorphism $f : \mathbb{Z}_6 \mapsto \mathbb{Z}_7^\times$ one should map generator to generator, then the corresponding powers or multiples must be matched. For instance, take $f([1]_6) = [3]_7$, then $f([n]_6) = {[3]_7}^n$, or more explicitly $f([2]_6) = [2]_7$, $f([3]_6) = [6]_7$, $f([4]_6) = [4]_7$, $f([5]_6) = [5]_7$, and $f([0]_6) = [1]_7$.

2. In S_4 consider $L = \{(1), (1\,3), (2\,4), (1\,3)(2\,4)\}$. Use $M = \{(1), (1\,3)\}$ to prove that invariance (or normality) of subgroups is not transitive and not induced by inclusion, that is in general

$$J \lhd H \lhd G \text{ or } J \lhd H \leq G \not\Rightarrow J \lhd G,$$

because in this case $J = M$, $H = L$, $G = S_4$, $M \lhd L$ and even if $L \lhd S_4$, then $M \ntriangleleft S_4$. You can follow the following steps.

(a) Assume $L \leq S_4$ and check if $L \lhd S_4$. You can use Theorem 0.6, for instance listing and comparing left and right cosets of L in S_4 until you find a "counter-example" to Theorem 0.6.

(b) Assume $M \leq L$ and check if $M \lhd L$. You can use Theorem 0.6, for instance listing and comparing left and right cosets of M in L until you find a "counter-example" to Theorem 0.6.

(c) Assume $M \leq S_4$ and check if $M \ntriangleleft S_4$. You can use Theorem 0.6, for instance listing and comparing left and right cosets of M in S_4 until you find a "counter-example" to Theorem 0.6.

Solution First let's list the 24 elements of S_4:

$(1), (1\,2), (1\,3), (1\,4), (2\,3), (2\,4), (3\,4)$	these fix more than one element,
$(1\,2\,3), (1\,3\,2), (1\,2\,4), (1\,4\,2), (1\,3\,4), (1\,4\,3), (2\,3\,4), (2\,4\,3)$	these fix only one element,
$(1\,2\,3\,4), (1\,2\,4\,3), (1\,3\,2\,4), (1\,3\,4\,2), (1\,4\,3\,2), (1\,4\,2\,3)$	cycles which do not fix any element,
$(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)$	non-cycles which do not fix any element.

(a) Check if $L \lhd S_4$. We use Theorem 0.6 property (2), where now we have $G = S_4$ and $H = L$. We can list all the left L-cosets: these are exactly $\frac{|S_4|}{|L|} = \frac{24}{4} = 6$. Moreover, because those in Proposition 0.5 are equivalence relations, then b is in the equivalence class aL if and only if $bL = aL$. This means that (specifically in our case)

$$aL = \{a, b, c, d\} \Leftrightarrow aL = bL = cL = dL,$$

therefore we do not have to actually compute any of the cosets bL, cL, or dL, but only check if $aL = La$.

- $L = (1)L = (1\,3)L = (2\,4)L = (1\,3)(2\,4)L$, the right-cosets would be the same.
- $(1\,2)L = \{(1\,2), (1\,2)(1\,3), (1\,2)(2\,4), (1\,2)(1\,3)(2\,4)\} = \{(1\,2), (1\,3\,2), (1\,2\,4), (1\,3\,2\,4)\}$ $= (1\,3\,2)L = (1\,2\,4)L = (1\,3\,2\,4)L$. Let's compute the right coset by $(1\,2)$: $L(1\,2) = \{(1\,2), (1\,3)(1\,2), (2\,4)(1\,2), (1\,3)(2\,4)(1\,2)\} = \{(1\,2), (1\,2\,3), (1\,4\,2), (1\,4\,2\,3)\} \neq (1\,2)L$, therefore $L \ntriangleleft S_4$ and we could stop. Therefore the conclusion is that $L \ntriangleleft S_4$ and there is no need to check transitivity or invariance.

For the sake of curiosity, we obtain the partition $\frac{S_4}{\mathcal{L}_L}$ by computing the remaining four left cosets.

- $(1\,4)L = \{(1\,4), (1\,4)(1\,3), (1\,4)(2\,4), (1\,4)(1\,3)(2\,4)\} = \{(1\,4), (1\,3\,4), (1\,4\,2), (1\,3\,4\,2)\}$ $= (1\,3\,4)L = (1\,4\,2)L = (1\,3\,4\,2)L$;
- $(2\,3)L = \{(2\,3), (2\,3)(1\,3), (2\,3)(2\,4), (2\,3)(1\,3)(2\,4)\} = \{(2\,3), (1\,2\,3), (2\,4\,3), (1\,2\,4\,3)\}$ $= (1\,2\,3)L = (2\,4\,3)L = (1\,2\,4\,3)L$;
- $(3\,4)L = \{(3\,4), (3\,4)(1\,3), (3\,4)(2\,4), (3\,4)(1\,3)(2\,4)\} = \{(3\,4), (1\,4\,3), (2\,3\,4), (1\,4\,2\,3)\}$ $= (1\,4\,3)L = (2\,3\,4)L = (1\,4\,2\,3)L$;
- $(1\,2\,3\,4)L = \{(1\,2\,3\,4), (1\,2\,3\,4)(1\,3), (1\,2\,3\,4)(2\,4), (1\,2\,3\,4)(1\,3)(2\,4)\} = \{(1\,2\,3\,4), (1\,4)(2\,3), (1\,2)(3\,4), (1\,4\,3\,2)\} = (1\,4)(2\,3)L = (1\,2)(3\,4)L = (1\,4\,3\,2)L$.

We can write down the quotient set

$$\frac{S_4}{\mathcal{L}_L} = \{L, (1\,2)L, (1\,4)L, (2\,3)L, (3\,4)L, (1\,4)(2\,3)L\}$$

and we notice that, for instance, $(1\,4)(2\,3)L = (1\,2)(3\,4)L$, while $(1\,2)L \odot (1\,4)L = ((1\,2)(1\,4))\,L = (1\,4\,2)L = (1\,4)L$. But from the computations above,
we can see that $(1\,2)L = (1\,3\,2)L$, while $(1\,3\,2)L \odot (1\,4)L = ((1\,3\,2)(1\,4))\,L = (1\,4\,3\,2)L = (1\,4)(2\,3)L$, therefore the products $(1\,2)L \odot (1\,4)L$ and $(1\,3\,2)L \odot (1\,4)L$ are not the same even if the factors are the same elements of $\frac{S_4}{\mathcal{L}_L}$: this product depends on the representatives.Therefore the induced product $\odot$ on $\frac{S_4}{\mathcal{L}_L}$ *does not define a group*! We had to expect this because L is not normal in S_4.

(b) We proceed as in part (2a). Now $G = L = \{(1), (1\,3), (2\,4), (1\,3)(2\,4)\}$ and $H = M = \{(1), (1\,3)\}$. We must list and compare left and right cosets of M in L, that is cosets of the type gM for $g \in L$. As above, these are going to be $\frac{|L|}{|M|} = \frac{4}{2} = 2$.

- $M = (1)M = (1\,3)M = M(1\,3)$.
- $(1\,3)(2\,4)M = \{(1\,3)(2\,4), (1\,3)(2\,4)(1\,3)\} = \{(1\,3)(2\,4), (2\,4)\} = (2\,4)M$, and $M(1\,3)(2\,4) = \{(1\,3)(2\,4), (1\,3)(1\,3)(2\,4)\} = \{(1\,3)(2\,4), (2\,4)\} = (1\,3)(2\,4)M$ ✓

Therefore $M \lhd L$.

(c) In order to prove that $M \ntriangleleft S_4$, we can still use Theorem 0.6 property (2), for $G = S_4$ and $H = M$. We only need to find one element $g \in S_4$ such that $gH \neq Hg$. Let's use, for instance, $g = (1\,2\,3)$:

$$(1\,2\,3)M = \{(1\,2\,3), (1\,2\,3)(1\,3)\} = \{(1\,2\,3), (2\,3)\}$$
$$M(1\,2\,3) = \{(1\,2\,3), (1\,3)(1\,2\,3)\} = \{(1\,2\,3), (1\,2)\} \neq (1\,2\,3)M \quad ✓$$

3. The only (up to isomorphism) non-cyclic group of order 4 is Klein 4-group K (see example 3.3.3, page 119). $K = \{1, i, j, k\}$ has multiplication table

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	1	k	j
j	j	k	1	i
k	k	j	i	1

Note: $i^2 = j^2 = k^2 = 1$

Is K isomorphic to the group L in Exercise 2? Justify your answer by either providing an isomorphism or arguments against the existence of an isomorphism.

Solution For $L = \{(1), (1\,3), (2\,4), (1\,3)(2\,4)\}$ one can compute

$$(1\,3)^2 = (2\,4)^2 = (1\,3)(2\,4)^2 = (1),$$

therefore L has all non-neutral elements of order 2, then L is a non-cyclic group of order 4 and it must be isomorphic to K. One isomorphism is $f : K \mapsto L$ such that $f(1) = (1)$, $f(i) = (1\,3)$, $f(j) = (2\,4)$, and $f(k) = (1\,3)(2\,4)$.

4. Consider again Klein 4-group from Exercise 3. In example 3.3.3 at page 119, $\mathbb{Z}_2 \times \mathbb{Z}_2$ denotes the direct product $(\mathbb{Z}_2, +) \times (\mathbb{Z}_2, +)$, which is an additive copy of K.
Following Proposition 3.3.4 at page 118, the direct product of two groups $(G, *)$ and $(F, \odot)$ is defined as the algebraic structure over the cartesian product $G \times F$ with *component-by-component* operation $\circledast$ such that

$$(a_1, b_1) \circledast (a_2, b_2) = (a_1 * a_2, b_1 \odot b_2)\,.$$

Write the multiplication table of the direct product $\mathbb{Z}_2 \times \mathbb{Z}_4$, using the notation 0 and 1 for the first components, and $\bar{a} = [a]_4$ for the second ones, then starting the table with the pairs $(0, \bar{0})$ and $(0, \bar{2})$.

Solution Actually in this case the factor structures are both additive and we can talk about the *addition table* of a *direct sum*. The order is $|\mathbb{Z}_2 \times \mathbb{Z}_4| = |\mathbb{Z}_2| \cdot |\mathbb{Z}_4| = 2 \cdot 4 = 8$. In particular, both factor structures are abelian, therefore the direct sum will be abelian and this group *cannot be isomorphic to* Q, the Quaternion group.

$+$	$(0,\bar{0})$	$(0,\bar{2})$	$(0,\bar{1})$	$(0,\bar{3})$	$(1,\bar{0})$	$(1,\bar{2})$	$(1,\bar{1})$	$(1,\bar{3})$
$(0,\bar{0})$	$(0,\bar{0})$	$(0,\bar{2})$	$(0,\bar{1})$	$(0,\bar{3})$	$(1,\bar{0})$	$(1,\bar{2})$	$(1,\bar{1})$	$(1,\bar{3})$
$(0,\bar{2})$		$(0,\bar{0})$	$(0,\bar{3})$	$(0,\bar{1})$	$(1,\bar{2})$	$(1,\bar{0})$	$(1,\bar{3})$	$(1,\bar{1})$
$(0,\bar{1})$			$(0,\bar{2})$	$(0,\bar{0})$	$(1,\bar{1})$	$(1,\bar{3})$	$(1,\bar{2})$	$(1,\bar{0})$
$(0,\bar{3})$				$(0,\bar{2})$	$(1,\bar{3})$	$(1,\bar{1})$	$(1,\bar{0})$	$(1,\bar{2})$
$(1,\bar{0})$					$(0,\bar{0})$	$(0,\bar{2})$	$(0,\bar{1})$	$(0,\bar{3})$
$(1,\bar{2})$						$(0,\bar{0})$	$(0,\bar{3})$	$(0,\bar{1})$
$(1,\bar{1})$							$(0,\bar{2})$	$(0,\bar{0})$
$(1,\bar{3})$								$(0,\bar{2})$

The table above completes this exercise.
For curiosity's sake, let's look at the order of these elements, knowing that the neutral element is $(0,\bar{0})$.

$$2(0,\bar{2}) = 2(1,\bar{0}) = 2(1,\bar{2}) = (0,\bar{0}) \Rightarrow \circ(0,\bar{2}) = \circ(1,\bar{0}) = \circ(1,\bar{2}) = 2,$$
$$2(0,\bar{1}) = 2(0,\bar{3}) = 2(1,\bar{1}) = (0,\bar{2}) \Rightarrow \circ(0,\bar{1}) = \circ(1,\bar{1}) = \circ(0,\bar{2}) = 4.$$

Therefore this group is not cyclic.

5. Consider the (multiplicative) Quaternion group $Q = \{1, -1, i, j, k, -i, -j, -k\}$ (see example 3.3.7, page 122).
   (a) What is the order of Q?
   (b) Provide a subgroup of order 6 in Q if possible (justify your answer).
   (c) Provide a subgroup of order 4 in Q if possible (justify your answer).
   (d) Provide a subgroup of order 2 in Q if possible (justify your answer).
   (e) Provide an **additive copy** $(G,+)$ of Q with the corresponding table of operations and isomorphism if possible (justify your answer).

Solution The multiplication table of Q is

$\cdot$	1	-1	i	j	k	$-i$	$-j$	$-k$
1	1	-1	i	j	k	$-i$	$-j$	$-k$
-1	-1	1	$-i$	$-j$	$-k$	i	j	k
i	i	$-i$	-1	k	$-j$	1	$-k$	j
j	j	$-j$	$-k$	-1	i	k	1	$-i$
k	k	$-k$	j	$-i$	-1	$-j$	i	1
$-i$	$-i$	i	1	$-k$	j	-1	k	$-j$
$-j$	$-j$	j	k	1	$-i$	$-k$	-1	i
$-k$	$-k$	k	$-j$	i	1	j	$-i$	-1

Note: $i^2 = j^2 = k^2 = -1$
$i^3 = -i, j^3 = -j, k^3 = -k$
$i^4 = j^4 = k^4 = 1$
$(-1)^2 = 1$

(a) The order of Q is 8 (the number of its elements).
(b) By Lagrange (Theorem 0.3), there isn't any subgroup of order 6 in Q because 6 doesn't divide 8.
(c) $\langle i \rangle = \{i, -1, -i, 1\}$ is a subgroup of order 4 in Q.
(d) $\langle -1 \rangle = \{-1, 1\}$ is a subgroup of order 2 in Q.

(e) Define $(G,+)$ with $G=\{0,\bar{0},a,\bar{a},b,\bar{b},c,\bar{c}\}$ and the operation with additive table such that $2\,\bar{0}=0$, $2\,a=2\,b=2\,c=\bar{0}$, $a+b=c$, and $\bar{a}+\bar{b}=\bar{c}$. Moreover, $3\,x=\bar{0}+x=\bar{x}$ for $x\in\{a,b,c\}$, and $4\,a=4\,b=4\,c=0$:

$+$	0	$\bar{0}$	a	b	c	$\bar{a}$	$\bar{b}$	$\bar{c}$
0	0	$\bar{0}$	a	b	c	$\bar{a}$	$\bar{b}$	$\bar{c}$
$\bar{0}$	$\bar{0}$	0	$\bar{a}$	$\bar{b}$	$\bar{c}$	a	b	c
a	a	$\bar{a}$	$\bar{0}$	c	$\bar{b}$	0	$\bar{c}$	b
b	b	$\bar{b}$	$\bar{c}$	$\bar{0}$	a	c	0	$\bar{a}$
c	c	$\bar{c}$	b	$\bar{a}$	$\bar{0}$	$\bar{b}$	a	0
$\bar{a}$	$\bar{a}$	a	0	$\bar{c}$	b	$\bar{0}$	c	$\bar{b}$
$\bar{b}$	$\bar{b}$	b	c	0	$\bar{0}$	$\bar{c}$	$\bar{0}$	a
$\bar{c}$	$\bar{c}$	c	$\bar{b}$	a	0	b	$\bar{a}$	$\bar{0}$

An isomorphism $f:Q\mapsto G$ must have $f(1)=0$ and $f(-1)=\bar{0}$, then we could choose, for instance, $f(i)=a$ and $f(j)=b$, then all the other mappings must follow according to the multiplication and additive tables: in this case it must be $f(-i)=\bar{a}$ and so on.

6. Using isomorphisms we can classify groups, that is provide "standard copies" of groups with given order. For instance any group of order a prime number p is isomorphic to $(\mathbb{Z}_p,+)$, while other are isomorphic to Klein's 4-group, the Quaternion group, and so on. According to Cayley's Theorem any group G is isomorphic to a subgroup of a symmetric group.

   Classify the multiplicative group $G=\left(\mathbb{Z}_8^\times,\cdot\right)$, that is find

   (a) the order of G,

   (b) the multiplication table of G and the order of its elements, and

   (c) a known (or standard) isomorphic copy of G with an isomorphism.

Solution $\mathbb{Z}_8^\times$ is the group of units in $\mathbb{Z}_8$, that is the congruence classes $[a]_8=\bar{a}$ with representative a coprime with 8. Then $\mathbb{Z}_8^\times=\{\bar{1},\bar{3},\bar{5},\bar{7}\}$.

(a) The order of G is 4.

(b) Because G is abelian, only the upper-triangular part of the multiplication table must be reported

$\cdot$	$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{7}$
$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{7}$
$\bar{3}$		$\bar{1}$	$\bar{7}$	$\bar{5}$
$\bar{5}$			$\bar{1}$	$\bar{3}$
$\bar{7}$				$\bar{1}$

Note: $\bar{3}^2=\bar{5}^2=\bar{7}^2=\bar{1}$

in particular there isn't any element with order 4 and G is not cyclic.

(c) The only (up to isomorphisms) non-cyclic group of order 4 is Klein 4-group (see exercise 3). Therefore G is isomorphic to K and an isomorphism $f:K\to G$ is given by $f(1)=\bar{1}$, $f(i)=\bar{3}$, $f(j)=\bar{5}$, and consequently $f(k)=f(ij)=f(i)f(j)=\bar{3}\,\bar{5}=\bar{7}$.