Math 310-010 - Spring 2014 - Test 2 - Part 1

Instructor: Dr. Francesco Strazzullo My Name KEY

Instructions. SHOW YOUR WORK neatly, please. Each exercise is worth 10 points. If using a result from the book, make a reference to it (using the page number as well).

1. Provide a non-trivial partition $\mathcal{P}$ of the set $S = \{1, a, 3, b, \pi\}$ and show that $\mathcal{P}$ is a partition of S.

P IS A PARTITION OF S IF I) "$X \in P \Rightarrow X \subseteq S$"; II) "$X, Y \in P$ WITH $X \neq Y \Rightarrow X \cap Y = \emptyset$"; AND III) $S = \bigcup_{X \in P} X$.

FOR EXAMPLE $P = \{\{1, 3, \pi\}, \{a, b\}\}$ IS A PARTITION WITH ONLY TWO ELEMENTS $X = \{1, 3, \pi\}$ AND $Y = \{a, b\}$ THAT SATISFY I), II) AND III)

2. Form the product table of the set of permutations $S = \{(123), (12), (14)\}$ with respect to the composition of functions in S_4.

$(123)(123) = (132)$; $(123)(14) = (1423)$; $(14)(123) = (1234)$
$(123)(12) = (13)$; $(12)(123) = (23)$; $(12)(14) = (142)$;
$(14)(12) = (124)$; $(12)(12) = (1)$; $(14)(14) = (1)$

	(12)	(14)	(123)
(12)	(1)	(142)	(23)
(14)	(124)	(1)	(1234)
(123)	(13)	(1423)	(132)

3. Check if the relation defined in $\mathbb{R}^2$ by

$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow x_1 + y_1 = x_2 + y_2,$$

is an equivalence. If $\sim$ is an equivalence then **represent graphically four distinct equivalence classes** $[(x, y)]_\sim$.

i) REFLEXIVE: $X+Y = X+Y \Rightarrow (X,Y) \sim (X,Y)$ ✓

ii) SYMMETRIC: $(X_1,Y_1) \sim (X_2,Y_2) \Rightarrow X_1+Y_1 = X_2+Y_2 \Rightarrow$

$\Rightarrow X_2+Y_2 = X_1+Y_1 \Rightarrow (X_2,Y_2) \sim (X_1,Y_1)$ ✓

iii) TRANSITIVE: $\left.\begin{array}{l}(X_1,Y_1) \sim (X_2,Y_2) \Rightarrow X_1+Y_1 = X_2+Y_2 \\ (X_2,Y_2) \sim (X_3,Y_3) \Rightarrow X_2+Y_2 = X_3+Y_3\end{array}\right] \Rightarrow$ ✓

$\Rightarrow X_1+Y_1 = X_2+Y_2 = X_3+Y_3 \Rightarrow X_1+Y_1 = X_3+Y_3 \Rightarrow (X_1,Y_1) \sim (X_3,Y_3)$

*iv) EQUIVALENCE CLASSES: SAY $X_1+Y_1 = C_1$ THEN

"$(X,Y) \sim (X_1,Y_1) \Leftrightarrow X+Y = C_1$" THEREFORE

$[(X_1,Y_1)]_\sim = \{(X,Y) \in \mathbb{R}^2 \mid X+Y = C_1\}$ = "STRAIGHT LINE $Y = -X + C_1$"

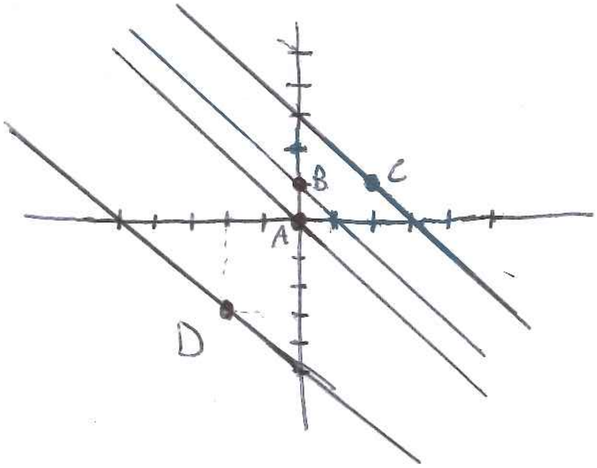
NOTE THAT $[(X_i,Y_i)]_\sim$ AND $[(X_2,Y_2)]_\sim$ ARE ALL PARALLEL TO THE MAIN ANTIDIAGONAL $[(0,0)]_\sim$ WITH EQUATION $Y = -X$.

A: $[(0,0)]_\sim$: $Y = -X$

B: $[(0,1)]_\sim$: $Y = -X + (0+1) = -X + 1$

C: $[(2,1)]_\sim$: $Y = -X + (2+1) = -X + 3$

D: $[(-2,-3)]_\sim$: $Y = -X + (-2+3) = -X - 5$



Math 310-010 - Spring 2014 - Test 2 - Part 2

Instructor: Dr. Francesco Strazzullo My Name____________________________

I certify that I did not receive third party help in completing this test. (*sign*)____________________________

Instructions. SHOW YOUR WORK neatly, please. Each exercise is worth 10 points. If using a result from the book, make a reference to it (using the page number as well).

1. Let $\mathbb{Z}^* = \mathbb{Z} - \{0\}$ and define $\mathbb{Q}$ to be the quotient set of $\mathbb{Z} \times \mathbb{Z}^*$ by the following equivalence relation

$$(a_1, b_1) \sim (a_2, b_2) \Leftrightarrow a_1 b_2 = b_1 a_2.$$

The elements of $\mathbb{Q}$ (i.e. the *rational numbers*) are equivalence classes $[(a, b)]_\sim$. These are usually denoted by $\frac{a}{b}$ and one calls (a, b) a representative of the rational number $\frac{a}{b}$. Moreover, $\frac{a}{b}$ can always be written in lowest terms, that is a and b are coprime, or relatively prime. For instance $(1, 2)$ and $(-3, -6)$ are distinct representatives of the same rational number $\frac{1}{2}$.
Check if the function $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$

$$f\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) = \frac{a_1 + a_2}{b_1 + b_2},$$

is well defined, that is check if $f\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right)$ depends on the representatives.

2. Check if the relation defined in $\mathbb{R}^2$ by

$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2,$$

is an equivalence. If $\sim$ is an equivalence then **represent graphically four distinct equivalence classes** $[(x, y)]_\sim$.

3. Consider the function $f : \mathbb{Z} \to \mathbb{Z}_{15}$ defined by

$$f(x) = [21x]_{15}$$

Describe the equivalence relation $\sim_f$, the quotient set $\frac{\mathbb{Z}}{f}$, and a bijective map between $\frac{\mathbb{Z}}{f}$ and $f(\mathbb{Z})$.

4. Write a disjoint-cycles-decomposition of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 8 & 3 & 5 & 7 & 4 & 1 & 6 \end{pmatrix},$$

then compose the multiplication table of these cycles.

MAT 320 - EXAM 2 PART 2 - SPRING 2014
KEY

1) IT IS ENOUGH TO PROVIDE A "NEGATIVE EXAMPLE". TO SHOW THAT:

$$(x_1, y_1) = (x_2, y_2) \not\Rightarrow f(x_1, y_1) = f(x_2, y_2)$$

HERE $(x_1, y_1) = (x_2, y_2) \Leftrightarrow x_1 = x_2, y_1 = y_2 \Leftrightarrow$

$$\Leftrightarrow \frac{a_1}{b_1} = x_1 = x_2 = \frac{c_1}{d_1} \text{ AND } \frac{a_2}{b_2} = y_1 = y_2 = \frac{c_2}{d_2}.$$

COUNTER-EXAMPLE: $\frac{1}{2} = \frac{3}{6} = x_1$, $\frac{2}{3} = \frac{4}{6} = y_1$

$$\left.\begin{aligned} f(x_1, y_1) &= f\left(\frac{1}{2}, \frac{2}{3}\right) = \frac{1+2}{2+3} = \frac{3}{5} \\ f(x_2, y_2) &= f\left(\frac{3}{6}, \frac{4}{6}\right) = \frac{3+4}{6+6} = \frac{7}{12} \end{aligned}\right] \text{ BUT } \frac{3}{5} \neq \frac{7}{12}$$

$$\text{NOTE: } \left.\begin{aligned} f\left(\frac{1}{2}, \frac{3}{6}\right) &= \frac{1+3}{2+6} = \frac{4}{8} = \frac{1}{2} \\ f\left(\frac{5}{10}, \frac{21}{42}\right) &= \frac{5+21}{10+42} = \frac{26}{52} = \frac{1}{2} \end{aligned}\right] \checkmark \text{ IT WORKS!}$$

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2) RIFLEXIVITY: $x^2 + y^2 = x^2 + y^2 \Rightarrow (x, y) \sim (x, y)$ ✓

SYMMETRY: $(x_1, y_1) \sim (x_2, y_2) \Rightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2 \Rightarrow$

$$\Rightarrow x_2^2 + y_2^2 = x_1^2 + y_1^2 \Rightarrow (x_2, y_2) \sim (x_1, y_1)$$

TRANSITIVITY:

$$\left.\begin{aligned} (x_1, y_1) \sim (x_2, y_2) &\Rightarrow \overbrace{x_1^2 + y_1^2}^{c_1} = \overbrace{x_2^2 + y_2^2}^{c_2} \\ (x_2, y_2) \sim (x_3, y_3) &\Rightarrow \underbrace{x_2^2 + y_2^2}_{c_2} = \underbrace{x_3^2 + y_3^2}_{c_3} \end{aligned}\right] \Rightarrow c_1 = c_2 = c_3 \Rightarrow$$

OVER $\Rightarrow$

$\left.\begin{matrix}2\\ \text{CONT}\end{matrix}\right) \Rightarrow C_1 = C_3 \Rightarrow x_1^2 + y_1^2 = x_3^2 + y_3^2 \Rightarrow (x_1, y_1) \sim (x_3, y_3)$ ✓

SAY (x_1, y_1) IS A FIXED PAIR WITH $x_1^2 + y_1^2 = C$ (CONSTANT)

THEN:

$$[(x_1, y_1)]_\sim = \{(x,y) \in \mathbb{R} \mid x^2 + y^2 = C\} =$$

= "CIRCLE WITH CENTER THE ORIGIN (CENTRAL CIRCLE) AND RADIUS $\sqrt{C}$ "

[NOTE, EQ. OF CIRCLE WITH CENTER (h, k) AND RADIUS R:

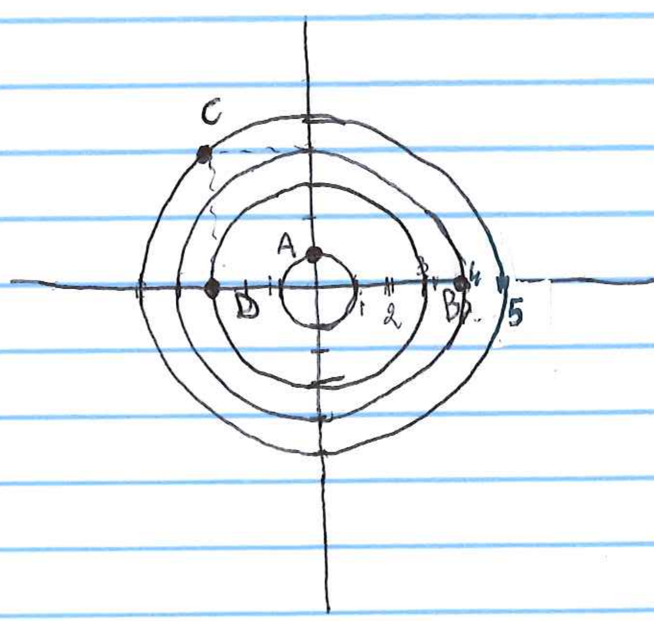
$$(x-h)^2 + (y-k)^2 = R^2$$

$[\overset{A}{(0,1)}]_\sim$ = "CENTRAL CIRCLE WITH RADIUS $\sqrt{0^2+1^2} = 1$

$[\overset{B}{(4,0)}]_\sim$ = "CENTRAL CIRCLE WITH RADIUS $\sqrt{4^2+0^2} = 4$

$[\overset{C}{(-3,4)}]_\sim$ = "CENTRAL CIRCLE WITH RADIUS $\sqrt{(-3)^2+4^2} = 5$

$[\overset{D}{(-3,0)}]_\sim$ = "CENTRAL CIRCLE WITH RADIUS $\sqrt{(-3)^2+0^2} = 3$



3) $f: \mathbb{Z} \to \mathbb{Z}_{15}$, $f(x) = [21x]_{15}$ (WE ARE NOT ASKED TO CHECK THAT f IS WELL DEFINED!)

LET $x_1, x_2 \in \mathbb{Z}$:

$x_1 \sim_f x_2 \Leftrightarrow f(x_1) = f(x_2) \Leftrightarrow [21x_1]_{15} = [21x_2]_{15} \Leftrightarrow$

$\Leftrightarrow 21x_1 - 21x_2 = 15 \cdot q \Leftrightarrow 3 \cdot 7(x_1 - x_2) = 3 \cdot 5 \cdot q \Leftrightarrow$ (CANCELLATION LAW IN $\mathbb{Z}$)

$\Leftrightarrow 7(x_1 - x_2) = 5q \Leftrightarrow 7x_1 - 7x_2 = 5q \Leftrightarrow [7x_1]_5 = [7x_2]_5$

$\Leftrightarrow [7]_5 \cdot [x_1]_5 = [7]_5 \cdot [x_2]_5 \Leftrightarrow$ ($[7]_5 = [2]_5$ IS NOT A DIVISOR OF ZERO THEN CANCELLATION LAW IN $\mathbb{Z}_5$ APPLIES.)

$\Leftrightarrow [x_1]_5 = [x_2]_5$ THEREFORE $\sim_f$ IS $\equiv_5$ (CONGRUENCE MOD 5) AND $\mathbb{Z}/_f = \mathbb{Z}/_{\equiv_5} = \mathbb{Z}_5$. WE DESCRIBED $\sim_f$ AND $\mathbb{Z}/_f$.

A BIJECTIVE MAP DEFINED IN 2.2.7 (pg 68) IS: $\bar{f}: \mathbb{Z}_f \to f(\mathbb{Z})$ SUCH THAT

$\bar{f}([x]_{\sim_f}) = f(x)$. IN THIS CASE: $\bar{f}: \mathbb{Z}_5 \to f(\mathbb{Z})$

AND $\bar{f}([x]_5) = [21x]_{15}$ *NOTE AT BOTTOM PAGE: $f(\mathbb{Z}) = \{[0]_{15}, [3]_{15}, [6]_{15}, [9]_{15}, [12]_{15}\}$

4) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 8 & 3 & 5 & 7 & 4 & 1 & 6 \end{pmatrix} = (1\,2\,8\,6\,4\,5\,7)$ ONLY ONE 7-CYCLE

THE MULTIPLICATION TABLE IS JUST $(1286457)(1286457) =$
$= (1847265)$

* $\bar{f}([0]_5) = [0]_{15}$; $\bar{f}([1]_5) = [21]_{15} = [6]_{15}$; $\bar{f}([2]_5) = [42]_{15} = [12]_{15}$
$\bar{f}([3]_5) = [63]_{15} = [3]_{15}$; $\bar{f}([4]_5) = [84]_{15} = [9]_{15}$.