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Math 221- Fall 2012 - Test 2

Instructor: Dr. Francesco Strazzullo Name KEY

Instructions. You are expected to use a graphing calculator or a software to complete some problems. You can use our class notes or alternatively a maximum of 4 size-letter sheets of notes. Sketch any graph that you use or tables of input/output, **approximating up to the fourth decimal place**. Each problem is worth 10 points.
SHOW YOUR WORK NEATLY, PLEASE.

1. Compute the first derivatives of the following functions. Show your work, applying the correct derivation rule, and give a simplified expression. Each part is worth 10 points.

(a) $y = \ln\left(\frac{(1+3x)^4}{\sqrt{2x+2}}\right)$

$$Y = \ln(1+3X)^4 - \ln(2X+2)^{\frac{1}{2}} = 4\ln(1+3X) - \frac{1}{2}\ln(2X+2)$$

$$Y' = 4\frac{d}{dX}[\ln(1+3X)] - \frac{1}{2}\frac{d}{dX}[\ln(2X+2)] = 4\frac{3}{1+3X} - \frac{1}{2}\frac{2}{2X+2}$$

$u = 1+3X$, $u' = 3$

$u = 2X+2$, $u' = 2$

$$\frac{d}{dX}[\ln u] = \frac{u'}{u}$$

$$Y' = \frac{12}{1+3X} - \frac{1}{2X+2} = \frac{12(2X+2)-(1+3X)}{(1+3X)(2X+2)} = \frac{24X+24-1-3X}{(1+3X)(2X+2)}$$

$$Y' = \frac{21X+23}{(1+3X)(2X+2)}$$

(b) $y = \arccos(3x-4)$ $= \cos^{-1}(3X-4)$

$$\frac{d}{dX}[\cos^{-1} u] = \frac{-u'}{\sqrt{1-u^2}}$$

$u = 3X-4$, $u' = 3$

$$Y' = \frac{-(3)}{\sqrt{1-(3X-4)^2}}$$

$$Y' = \frac{-3}{\sqrt{1-(9X^2-24X+16)}} = \frac{-3}{\sqrt{-9X^2+24X-15}}$$

20)

2. Compute the **second order derivative** of $y = e^{2x} - \sin x$.

$$y'' = \frac{d}{dx}[y'] \quad ; \quad \frac{d}{dx}[e^u] = u' e^u \quad , \quad u = 2x \, , \quad u' = 2$$

$$y' = \frac{d}{dx}[e^{2x}] - \frac{d}{dx}[\sin x] = 2e^{2x} - \cos x$$

$$y'' = 2\frac{d}{dx}[e^{2x}] - \frac{d}{dx}[\cos x] = 2(2e^{2x}) - (-\sin x)$$

$$y'' = 4e^{2x} + \sin x$$

3. Using implicit differentiation, write the equation of the tangent line to graph of $x^3 - y^3 + 3y = -25$ at the point $(-3, -2)$.

$$\frac{d}{dx}[\underbrace{x^3}_{\text{SIMPLE POWER}} - \underbrace{y^3}_{\text{GEN POWER}} + 3y] = \frac{d}{dx}[-25] \rightarrow \frac{d}{dx}[x^3] - \frac{d}{dx}[y^3] + 3\frac{dy}{dx} = 0$$

$$\rightarrow 3x^2 - 3y^2 y' + 3y' = 0 \rightarrow 3(-y^2 + 1)y' = -3x^2 \rightarrow$$

NOT NEEDED

$$\rightarrow y' = \frac{-3x^2}{3(1-y^2)} = \frac{x^2}{y^2 - 1} \rightarrow y'\Big|_{\substack{x=-3 \\ y=-2}} = \frac{(-3)^2}{(-2)^2 - 1} = \frac{9}{3} = 3$$

$$\text{EQ. TAN. LINE: } L(x) = b + y'\Big|_{\substack{x=a \\ y=b}} (x - a)$$

$$L(x) = -2 + 3(x - (-3))$$

$$= -2 + 3x + 9$$

$$= 3x + 7$$

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4. Consider the function $f(x) = x^2 - 3x + 3$. Compute $f'(1)$ in the following ways:

(a) with the *definition by limit*;

(b) by differentiation rules;

(c) with your calculator.

(a) $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$; $a = 1$, $f(a) = f(1) = 1$,

$f(a+h) = f(1+h) = (1+h)^2 - 3(1+h) + 3 = 1 + 2h + h^2 - 3 - 3h + 3$

$= h^2 - h + 1$

DIRECT SUBS

$f'(1) = \lim_{h \to 0} \frac{h^2 - h + 1 - (1)}{h} = \lim_{h \to 0} \frac{\cancel{h}(h-1)}{\cancel{h}} \stackrel{+}{=} 0 - 1 = -1$

(b) $f'(x) = 2x - 3 \to f'(1) = 2(1) - 3 = -1$ ✓

(c) 2^{ND} + TRACE + 6, ENTER 1, $\frac{dY}{dx} = -1$ ✓

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5. A population of 360 bacteria is introduced into a culture. The number of bacteria is modeled by

$$P(t) = 180\left(2 - \frac{3t^3}{60+2t^3}\right),$$

where t is measured in hours. What is the limiting size (or *carrying capacity*) of this population?

"LIMITING SIZE" $= \lim_{t\to\infty} P(t) = \lim_{t\to\infty} 180\left(2 - \frac{3t^3}{60+2t^3}\right)$

NOT NEEDED

$= 180 \lim_{t\to\infty}\left(2 - \frac{3t^3}{60+2t^3}\right) = 180\left(2 - \lim_{t\to\infty}\frac{3t^3}{60+2t^3}\right)$

$\lim_{t\to\infty} \frac{3t^3}{60+2t^3} = 3\lim_{t\to\infty}\frac{\cancel{t^3}\,1}{\cancel{t^3}\left(\frac{60}{t^3}+2\right)} = \frac{3}{60\left(\lim_{t\to\infty}\frac{1}{t^3}\right)+2} =$

SAME DEGREE

QUOTIENT OF LEADING COEFFICIENTS $=$

$\frac{3}{60(0)+2} = \frac{3}{0+2} = \frac{3}{2}$

$\to$ LIMITING SIZE $= 180\left(2 - \frac{3}{2}\right) = 90$ BACTERIA.

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6. Find (if any) the vertical and the horizontal asymptotes of the function $f(x) = \dfrac{3\ln x}{1-\ln x}$.

V.A.) DOMAIN: $\ln x \to x > 0$

$1 - \ln x \neq 0 \to x \neq e$ → TWO CANDIDATES FOR V.A. $x = 0,\ x = e$

$1 - \ln x = 0 \to \ln x = 1 \to x = e \approx 2.78$

I) WE CAN'T TAKE $x \leq 0$:

$\lim_{x \to 0^+} f(x) \overset{\text{DIR. SUB}}{=} \dfrac{-\infty}{\infty}$ → SIMPLIFY

$$\lim_{x \to 0^+} \frac{3\ln x}{1-\ln x} = \lim_{x\to 0^+} \frac{\cancel{\ln x}\,(3)}{\cancel{\ln x}\left(\frac{1}{\ln x} - 1\right)} = \frac{3}{\left(\lim_{x\to 0^+}\frac{1}{\ln x}\right) - 1}$$

$$\approx \frac{3}{\frac{1}{-\infty} - 1} \approx \frac{3}{0-1} = -3 \to x = 0 \text{ NOT V.A.}$$

II) $\lim_{x \to e^-} \dfrac{3\ln x}{1-\ln x} \overset{\text{DIR SUB}}{=} \dfrac{3(1)}{0} \approx \infty \to x = e$ IS A V.A.

H.A.) BECAUSE $x > 0$ WE CAN'T CONSIDER "$x \to -\infty$".

$\lim_{x\to\infty} f(x) \overset{\text{DIR SUB}}{=} \dfrac{\infty}{-\infty}$ → SIMPLIFY, AS IN PART I:

$$\lim_{x\to\infty} f(x) = \frac{3}{\left(\lim_{x\to\infty}\frac{1}{\ln x}\right) - 1} \approx \frac{3}{\frac{1}{\infty} - 1} \approx \frac{3}{0-1}$$

$= 3 \to Y = 3$ IS A H.A.

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