Mat 320 – Fall 2013 –Exam3

Instructor: Dr. Francesco Strazzullo **Name** KEY

I certify that I did not receive third party help in *completing* this test (sign) ____________

Instructions. Technology is allowed on this exam, unless specifically limited. Each problem is worth 10 points, except number 7. When using technology describe which commands (or keys typed) you used or print out and attach your worksheet.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1) Let $H = \{a + bx^2 + cx^4 \in P_4 : a = -c\}$ be a subset of the real vector space of polynomials of degree at most 4. If H is a subspace of P_4 then provide one of its bases, otherwise show which property of subspaces H does not satisfy.

$$\left.\begin{array}{l} \vec{p} \in H \Leftrightarrow \vec{p} = a + bx^2 - ax^4 = a(1-x^4) + bx^2 \\ \vec{p}_1 = 1 - x^4,\ \vec{p}_2 = x^2 \in \mathbb{P}_4 \end{array}\right] \Rightarrow H = \text{Span}\{\vec{p}_1, \vec{p}_2\} \text{ IS}$$

A SUBSPACE OF $\mathbb{P}_4$. $\mathcal{E} = \{1, x, x^2, x^3, x^4\}$ STANDARD BASIS IN $\mathbb{P}_4$.

WE WANT TO PROVE THAT $\mathcal{B} = \{\vec{p}_1, \vec{p}_2\}$ IS LIN. INDEP, THUS A BASIS FOR H.

$$[\vec{p}_1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \text{ AND } \vec{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow [\mathcal{B}]_{\mathcal{E}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} \xrightarrow{\text{RANK}} 2 \quad \text{THEN ITS}$$

TWO COLUMNS ARE INDEPENDENT $\Rightarrow$ $\mathcal{B}$ IS LINEARLY IND.

2) Let $\mathcal{C} = \left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\right\}$ be a subset of the real vector space of the 2-by-2 matrices $M_{2,2}$.

(a) Use the standard basis $\mathcal{E} = \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$ to prove that $\mathcal{C}$ is linearly independent (Hint: for each $C_i \in \mathcal{C}$ consider the vector of components (or coordinate vector) $[C_i]_{\mathcal{E}}$.)

(b) Use part (a) to extend $\mathcal{C}$ to a basis for $M_{2,2}$.

$$\mathcal{E} = \{ E_{ij} \mid 1 \le i \le 2,\ 1 \le j \le 2,\ (E_{ij})_{ij} = 1 \text{ AND } (E_{ij})_{hk} = 0 \}$$

$$C_1 = 1E_{11} + 0E_{12} + 0E_{21} + 1E_{22} \Rightarrow [C_1]_{\mathcal{E}} = \begin{bmatrix}1\\0\\0\\1\end{bmatrix};$$

$$[C_2]_{\mathcal{E}} = \begin{bmatrix}0\\-1\\1\\0\end{bmatrix}; \quad [C_3]_{\mathcal{E}} = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}. \Rightarrow [\mathcal{C}]_{\mathcal{E}} = \begin{bmatrix}1 & 0 & 0\\0 & -1 & 1\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix} \Rightarrow$$

$$\Rightarrow \mathrm{RREF}([\mathcal{C}]_{\mathcal{E}}) = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

RANK = 3

$\Downarrow$

$\mathcal{C}$ LIN. INDEP.

TO MAKE IT I_4

$\Rightarrow$ THE FOURTH VECTOR OF $\mathcal{E}$ CAN BE USED TO COMPLETE $\mathcal{C}$ TO A BASIS.

$\mathcal{B} = \{C_1, C_2, C_3, E_{22}\}$ IS A COMPLETION OF $\mathcal{C}$ TO A BASIS OF M_{22}

3) Consider two bases $\mathcal{B} = \left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$ and $\mathcal{C} = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}\right\}$ of $\mathbb{R}^2$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and use it to compute $[\boldsymbol{x}]_{\mathcal{B}}$ and $[\boldsymbol{x}]_{\mathcal{C}}$ for $\boldsymbol{x} = \begin{bmatrix}1\\0\end{bmatrix}$.

$$[\mathcal{B}]_{\mathcal{E}} = \left[\,[\vec{b}_1]_{\mathcal{E}} \;\; [\vec{b}_2]_{\mathcal{E}}\,\right] = \begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix} \; ; \; [\mathcal{B}]^{\mathcal{E}} = \left([\mathcal{B}]_{\mathcal{E}}\right)^{-1} = \frac{1}{3}\begin{bmatrix}1 & 1\\ 1 & -2\end{bmatrix}$$

AND ① $[\vec{x}]_{\mathcal{B}} = [\mathcal{B}]^{\mathcal{E}}[\vec{x}]_{\mathcal{E}}$, ② $[\vec{x}]_{\mathcal{E}} = [\mathcal{B}]_{\mathcal{E}}[\vec{x}]_{\mathcal{B}}$

CHANGE-OF-COORDINATES MATRIX FROM $\mathcal{B}$ TO $\mathcal{C}$ IS $[\mathcal{C}]^{\mathcal{B}}$ SUCH THAT $[\vec{x}]_{\mathcal{C}} = [\mathcal{C}]^{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$

PROPERTIES: I) $[\mathcal{C}]^{\mathcal{B}} = [\mathcal{B}]_{\mathcal{C}} = \left[\,[\vec{b}_1]_{\mathcal{C}} \cdots [\vec{b}_n]_{\mathcal{C}}\,\right]$

II) $[\mathcal{C}]^{\mathcal{B}} = [\mathcal{C}]^{\mathcal{E}}[\mathcal{B}]_{\mathcal{E}}$

WE USE II: $[\mathcal{C}]_{\mathcal{E}} = \begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix} \Rightarrow [\mathcal{C}]^{\mathcal{E}} = \left(\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\right)^{-1} = \begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}$

$$\Rightarrow [\mathcal{C}]^{\mathcal{B}} = \begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\cdot\begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}2 & 1\\ 3 & 3\end{bmatrix}.$$

$$\begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{B}} = \frac{1}{3}\begin{bmatrix}1 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{C}} = [\mathcal{C}]^{\mathcal{B}}\begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}2 & 1\\ 3 & 3\end{bmatrix}\left(\frac{1}{3}\begin{bmatrix}1\\1\end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix}3\\6\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

CHECK: $\begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{C}} = [\mathcal{C}]^{\mathcal{E}}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$ ✓

4) Consider two bases $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ and $\mathcal{C} = \{\boldsymbol{c}_1, \boldsymbol{c}_2\}$ of a real vector space $\mathbb{V}$ such that

$$\boldsymbol{b}_1 = \boldsymbol{c}_1 - 3\boldsymbol{c}_2 \text{ and } \boldsymbol{b}_2 = -2\boldsymbol{c}_1 + 7\boldsymbol{c}_2.$$

Suppose that $\boldsymbol{x}$ is a vector in $\mathbb{V}$ such that $\boldsymbol{x} = 2\boldsymbol{b}_1 + \boldsymbol{b}_2$, that is $[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find $[\boldsymbol{x}]_{\mathcal{C}}$.

LOOK AT EXERCISE 3. I $[\mathcal{C}]^{\mathcal{B}} = \left[[\vec{b}_1]_{\mathcal{C}} \cdots [\vec{b}_n]_{\mathcal{C}} \right] = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}$

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow [\vec{x}]_{\mathcal{C}} = [\mathcal{C}]^{\mathcal{B}} [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

THEN $\vec{x} = \vec{c}_2$.

CHECK: $\vec{x} = 2\vec{b}_1 + \vec{b}_2 = 2(\vec{c}_1 - 3\vec{c}_2) + (-2\vec{c}_1 + 7\vec{c}_2)$

$= 2\vec{c}_1 - 6\vec{c}_2 - 2\vec{c}_1 + 7\vec{c}_2 = \vec{c}_2$ ✓

5) Evaluate the determinant of $A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 2 & 0 & 1 \end{bmatrix}$ by expanding along the second row, then the second column, and finally using technology.

2ND ROW: $\det A = (-1)^{2+2} \cdot 1 \begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} + (-1)^{2+3} \cdot 4 \cdot \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = -9 - 4(0) = -9$

2ND COLUMN: $\det A = (-1)^{2+2} \cdot 1 \begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} = -9$

CALCULATOR CONFIRMS $\det A = -9$ ✓

6) Let $A = \begin{bmatrix} -5 & -6 \\ 8 & 9 \end{bmatrix}$, $P = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$, and $Q = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$. Determine if P or Q diagonalize A. What is the relation between P and Q?

B DIAGONALIZES A IF $B^{-1}AB = D$ IS A DIAGONAL MATRIX.

$P^{-1} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} = Q$ (THEREFORE P AND Q ARE INVERSE OF EACH OTHER)

USED CALCULATOR

NOT A SCALING OF <4, 1>

$P^{-1}AP = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -5 & -6 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -26 & -21 \\ 41 & 33 \end{bmatrix} = \begin{bmatrix} -149 & -120 \\ 190 & 153 \end{bmatrix}$

COULD USE A CALCULATOR.

SCALING OF <1, -1>

SCALING OF <-3, 4>

NOT DIAGONAL.

$Q^{-1}AQ = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -5 & -6 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -9 \\ -1 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ ✓

Q DIAGONALIZES A: ITS COLUMNS ARE EIGENVECTORS OF A.

7) Consider the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ 2 & 1 & -1 \end{bmatrix}$. **Each part is worth 10 points.**

(a) Is $\begin{bmatrix} 1+\frac{2}{3}\sqrt{3} \\ -1-\frac{1}{3}\sqrt{3} \\ 0 \end{bmatrix}$ an eigenvector of A? Determine its corresponding eigenvalue, if it is an eigenvector.

(b) Determine a basis for the eigenspace for the eigenvalue $\lambda = 3$ of A.

(a) $$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1+\frac{2}{3}\sqrt{3} \\ -1-\frac{1}{3}\sqrt{3} \\ 0 \end{bmatrix} = \begin{bmatrix} 1+\frac{2}{3}\sqrt{3}-\left(-1-\frac{1}{3}\sqrt{3}\right) \\ -3-\sqrt{3} \\ 2+\frac{4}{3}\sqrt{3}-1-\frac{1}{3}\sqrt{3} \end{bmatrix} = \begin{bmatrix} 2+\sqrt{3} \\ -3-\sqrt{3} \\ 1+\sqrt{3} \end{bmatrix}$$

$\neq 0$

NOT EIGENVECTOR.

(b) $$A-3I = \begin{bmatrix} 1-3 & -1 & 0 \\ 0 & 3-3 & 2 \\ 2 & 1 & -1-3 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & .5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$\Rightarrow$ GEN SOL. OF $(A-3I)\vec{x} = \vec{0}$ IS $\vec{S} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2\begin{bmatrix} -.5 \\ 1 \\ 0 \end{bmatrix}$

THEN A BASIS FOR THIS EIGENSPACE IS $\left\{\begin{bmatrix} -.5 \\ 1 \\ 0 \end{bmatrix}\right\}$.

8) Compute the characteristic polynomial and then solve the characteristic equation for $A = \begin{bmatrix}1&0&5&-3\\0&1&4&1\\2&0&3&0\\1&0&0&2\end{bmatrix}$. You can use technology **only to check your results**.

CHAR. METRIX: $A-\lambda I = \begin{bmatrix}1-\lambda&0&5&-3\\0&1-\lambda&4&1\\2&0&3-\lambda&0\\1&0&0&2-\lambda\end{bmatrix}$

EVEN "+"

2ND COLUMN

3RD ROW

I) CHAR. POLYNOMIAL = $\det(A-\lambda I) \underset{p}{=} (1-\lambda)\begin{vmatrix}1-\lambda&5&-3\\2&3-\lambda&0\\1&0&2-\lambda\end{vmatrix} = (1-\lambda)\left(\begin{vmatrix}5&-3\\3-\lambda&0\end{vmatrix}\right.$

EVEN "+"

$\left. + (2-\lambda)\begin{vmatrix}1-\lambda&5\\2&3-\lambda\end{vmatrix}\right) = (1-\lambda)\Big((2-\lambda)\big((1-\lambda)(3-\lambda)-10\big)+3(3-\lambda)\Big)$

$=(1-\lambda)\Big((2-\lambda)(\lambda^2-4\lambda-7)+9-3\lambda\Big)=(1-\lambda)\Big(2\lambda^2-8\lambda-14-\lambda^3+4\lambda^2$

$+7\lambda+9-3\lambda\Big)=(1-\lambda)\left(-\lambda^3+6\lambda^2-4\lambda-5\right)$

II) EIGENVALUES: $P(1)=0 \Rightarrow \lambda=1$ EIGENVALUE.

$P(\lambda)=(\lambda-1)(\lambda^3-6\lambda^2+4\lambda+5)$

$P(5)=0 \Rightarrow$ SYNT. DIV. $\begin{array}{r|rrr|r} & 1 & -6 & 4 & 5 \\ 5 & & 5 & -5 & -5 \\ \hline & 1 & -1 & -1 & 0 \end{array}$ ✓ $\Rightarrow$

$\Rightarrow P(\lambda)=(\lambda-1)(\lambda-5)(\lambda^2-\lambda-1) = (\lambda-1)(\lambda-5)\left(\lambda-\frac{1}{2}-\frac{\sqrt5}{2}\right)\left(\lambda-\frac{1}{2}+\frac{\sqrt5}{2}\right)$

$\lambda^2-\lambda-1=0 \Rightarrow \lambda=\frac{1\pm\sqrt{1+4}}{2}=\frac{1\pm\sqrt5}{2}$

EIGENVALUES: $\lambda = 1, 5, \frac{1-\sqrt5}{2}, \frac{1+\sqrt5}{2}$

9) Find the eigenvalues and a basis of eigenvectors for the matrix $A = \begin{bmatrix} 6 & 0 & 0 \\ -4 & 4 & 2 \\ 2 & 1 & 5 \end{bmatrix}$, then check your result by diagonalizing A.

I) $\det(A-\lambda I) = \begin{vmatrix} 6-\lambda & 0 & 0 \\ -4 & 4-\lambda & 2 \\ 2 & 1 & 5-\lambda \end{vmatrix} = (6-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ 1 & 5-\lambda \end{vmatrix} =$

$= (6-\lambda)\left((4-\lambda)(5-\lambda)-2\right) = (6-\lambda)(\lambda^2-9\lambda+18) = 0$ $\lambda = 6$ QUADRATIC ALG. MUL. $= 2$

$\lambda^2 - 9\lambda + 18 = 0 \Rightarrow (\lambda-6)(\lambda-3) = 0$ $\lambda = 6$, $\lambda = 3$

II) a) EIGENVALUE $\lambda = 6$:

$A - 6I = \begin{bmatrix} 0 & 0 & 0 \\ -4 & -2 & 2 \\ 2 & 1 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & .5 & -.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{GEN SOL} = \begin{bmatrix} -.5x_2 + .5x_3 \\ x_2 \\ x_3 \end{bmatrix}$

$= \frac{1}{2}x_2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \frac{1}{2}x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \Rightarrow$ "EIGENVECTORS FOR $\lambda = 6$" $= \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$

b) EIGENVALUE $\lambda = 3$:

$A - 3I = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{GEN SOL} = \begin{bmatrix} 0 \\ -2x_3 \\ x_3 \end{bmatrix} =$

$= x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \Rightarrow$ "EIGENVECTOR FOR $\lambda = 3$" $= \left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

III) BASIS OF EIGENVECTORS: $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

IV) IF $P = \begin{bmatrix} -1 & 1 & 0 \\ 2 & 0 & -2 \\ 0 & 2 & 1 \end{bmatrix}$ THEN $P^{-1}AP = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ ✓ WITH CALCULATOR.