MAT 221 - Fall 2016 - Exam 2

Instructor: Dr. Francesco Strazzullo **Name** KEY

I did not receive third party help in completing this test. *Signature*__________________

Instructions. You are expected to use a graphing calculator or software to complete some problems. Files can be downloaded and uploaded to the Eagleweb Coursework page for this assignment. Upload files or sketch any graph that you use or tables of input/output, **approximating up to the fourth decimal place**. Each problem is worth 10 points, unless otherwise specified. Available 100 points. **SHOW YOUR WORK NEATLY, PLEASE.** (no work = no points)

1. Find the value of the limit $\lim_{t\to 0}\frac{\sin 6t}{\sin 4t}$. $= \lim_{t\to 0} \frac{\sin(6t)}{6t}\frac{6t}{4t}\frac{4t}{\sin(4t)} =$

$$= \frac{6}{4}\lim_{t\to 0}\frac{\sin(6t)}{6t}\cdot\lim_{t\to 0}\left(\frac{\sin(4t)}{4t}\right)^{-1}$$

$$\lim_{t\to 0} ct = 0 \Rightarrow \lim_{t\to 0} f(ct) = \lim_{x\to 0} f(x) \Rightarrow$$

$$\Rightarrow \lim_{t\to 0}\frac{\sin 6t}{\sin 4t} = \frac{6}{4}\lim_{x\to 0}\frac{\sin x}{x}\cdot\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^{-1}$$

$$= \frac{3}{2}(1)(1) = \frac{3}{2}$$

2. Find the value of the limit $\lim_{x\to 3}\left(\frac{1}{x-3}-\frac{6}{x^2-9}\right)$. $= \lim_{x\to 3}\frac{1}{x+3} = \frac{1}{3+3} = \frac{1}{6}$

SIMPLIFY: $\frac{1}{x-3}-\frac{6}{x^2-9} = \frac{x+3-6}{(x-3)(x+3)} = \frac{x-3}{(x-3)(x+3)}$

3. Find the linear approximation to $f(x) = e^{-3x+x^2}$ at $x = 0$.

$f(x) \approx L(x) = f(a) + f'(a)(x-a)$ HERE $a = 0$

$f'(x) = (-3+2x)e^{-3x+x^2} \Rightarrow f'(0) = -3$; $f(0) = 1$ THEN

$L(x) = 1 - 3x$

4. Find $\frac{dy}{dx}$ when $y = (\sin x)^{\cos x}$.

LOG. DIFFERENTIATION: $\ln y = \ln(\sin x)^{\cos x} = \cos x \cdot \ln(\sin x)$

$\frac{d}{dx}[\ln y] = \frac{d}{dx}[\cos x \ln(\sin x)]$

$\frac{y'}{y} = -\sin x \ln(\sin x) + \cos x \cdot \frac{\cos x}{\sin x}$

$y' = \left(\frac{\cos^2 x}{\sin x} - \sin x \ln(\sin x)\right) \cdot (\sin x)^{\cos x}$ OR

$= \cos^2 x \cdot (\sin x)^{\cos x - 1} - \ln(\sin x) \cdot (\sin x)^{\cos x + 1}$

5. Find the slope of the normal to the curve with parametric equations $x = t + t^2$, $y = t + e^t$ at the point (0, 1). Use symbolic notation, do not approximate.

SLOPE OF TANGENT LINE: $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ IF NOT ZERO THEN

SLOPE OF NORMAL LINE IS $-\left(\dfrac{dy}{dx}\right)^{-1} = -\dfrac{\frac{dx}{dt}}{\frac{dy}{dt}}$

POINT $\begin{cases} t+t^2=0 \Rightarrow t=0,\cancel{1} \\ t+e^t=1 \end{cases}$ ONLY $t=0$ WORKS.

$\left.\dfrac{dx}{dt}\right|_{t=0} = \left.1+2t\right|_{t=0} = 1$; $\left.\dfrac{dy}{dt}\right|_{t=0} = \left.1+e^t\right|_{t=0} = 2$.

NORMAL LINE AT $(0,1)$, $t=0$, IS $-\dfrac{1}{2}$

6. If $xy = 8$, find the simplified expression of $\dfrac{d^2y}{dx^2}$ and its value at the point $(-2, 4)$.

I) $\dfrac{d}{dx}[xy] = \dfrac{d}{dx}[8] \Rightarrow 1\cdot y + xy' = 0 \Rightarrow y' = -\dfrac{y}{x}$

$\Rightarrow 0 = \dfrac{d}{dx}[y + xy'] = y' + y' + xy'' = -2\dfrac{y}{x} + xy'' \Rightarrow$

$\Rightarrow y'' = \dfrac{d^2y}{dx^2} = 2\dfrac{y}{x^2}$

AT $P=(-2,4)$: $\left.\dfrac{d^2y}{dx^2}\right|_{\substack{x=-2\\y=4}} = 2\dfrac{4}{4} = 2$

7. Let $y = x^4 + x^2 + 1$, $x = 1$, and $dx = 1$. Find the value of the differential dy.

$dy = y' \, dx = (4x^3 + 2x)\, dx$ THEN

$dy = 6 \cdot (1) = 6$

8. Let $f(x) = \begin{cases} \sqrt{-2-x} & \text{if } x \le -2 \\ x & \text{if } -2 < x \le 2 \\ x^2 - 4x + 6 & \text{if } x > 2 \end{cases}$. Complete the following parts.

(a) $f(-2) = \sqrt{-2-(-2)} = 0$

(b) $\lim_{x \to -2} f(x) =$

(c) Is $f(x)$ continuous at -2? If not, can we remove this discontinuity?

(b) $\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \sqrt{-2-x} = 0$

$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} x = -2$

DIFFERENT $\Rightarrow$ $\lim_{x \to -2} f(x)$ IS UNDEFINED

(c) THIS IS A "GAP" DISCONTINUITY AND IT CAN NOT BE REMOVED.

9. Let $P(t) = \dfrac{20,000}{1 + 1999e^{-0.1t}}$ be the population of a bacteria colony at time t hours. (a) What is the limiting size L (or *carrying capacity*) of this population? (b) Approximately (to the nearest integer), after how many hours can you say that the population has reached its carrying capacity, that is $P = L - 1$?

a) $L = \lim_{t \to \infty} P(t) = \dfrac{20000}{1 + 1999 \lim_{t \to \infty} e^{-.1t}} = \dfrac{20000}{1} = 20000$ BACTERIA

$\lim_{x \to \infty} e^{-x} = 0$

b) SOLVE $P(t) = 19999 \Rightarrow 19999 + (19999)(1999)e^{-.1t} = 20000 \Rightarrow$

$\Rightarrow e^{-.1t} = (39978001)^{-1} \Rightarrow e^{.1t} = 39978001 \Rightarrow .1t = \ln(39978001)$

$\Rightarrow t = 10 \ln(39978001) \approx 175.04$

CHECK: $P(175) \approx 19998$; $P(176) \approx 19999$.

THEREFORE AFTER 176 HOURS.

10. Below are the sunrise and sunset times (in standard time) for Moorhead, Minnesota on the 21st of each month in 2007.

Sunrise and Sunset in Moorhead, Minnesota on the 21st of each month in 2007

Date	Day of Year	Sunrise	Sunset	Day Length (hours)
Jan 21	21	08:03	17:14	9.18
Feb 21	52	07:21	18:01	10.67
Mar 21	80	06:28	18:41	12.22
Apr 21	111	05:29	19:24	13.92
May 21	141	04:45	20:02	15.28
Jun 21	172	04:32	20:25	15.88
Jul 21	202	04:54	20:12	15.30
Aug 21	233	05:32	19:27	13.92
Sep 21	264	06:12	18:27	12.25
Oct 21	294	06:53	17:29	10.60
Nov 21	325	07:39	16:47	9.13
Dec 21	355	08:09	16:41	8.53

X

$\dfrac{Y}{60}$

(a) Find the rate of change in day length with respect to time (in minutes per day) THE TABLE IS IN HOURS!

(i) from February 21 through March 21.

$$ARC_1 = \frac{12.22 - 10.67}{80 - 52} \approx .0554 \text{ h/DAY} \approx 3.3214 \text{ min/DAY}$$

(ii) from March 21 through April 21.

$$ARC_2 = \frac{13.92 - 12.22}{111 - 80} \approx .0548 \text{ h/DAY} \approx 3.2903 \text{ min/DAY}$$

(b) Estimate the instantaneous rate of change in day length for March 21.

$$IRC \approx \frac{ARC_1 + ARC_2}{2} = \frac{3.3214 + 3.2903}{2} \approx 3.3059 \text{ min/DAY}$$

(c) Compute the 3rd degree polynomial regression and the trigonometric (sine) regression that best fit the data above, so that the day length (in minutes) is a function of the time (day of the year). Report the correlation coefficients (round to 4 decimal places) and state which regression line is the better model in this context.

POLY: $y = p(x) = .00003x^3 - .0291x^2 + 7.37349x + 364.29361$, $R^2 = .9463$

TRIG: $y = t(x) = 728.62548 + 216.1631 \sin(.01674x - 1.31188)$, $R^2 = .9993$

THE TRIG MODEL IS THE BEST STATISTICALLY AND QUALITATIVELY (IN THE CONTEXT).

(d) Use the best model from part (c) to estimate the rate of change of the day length for March 21 and compare it to the one you found it in part (b).

$t'(80) \approx 3.6177$ min/DAY THEN THE ESTIMATE IRC IS GOOD TO THE INTEGRAL PART (NOT TO THE NEAREST INTEGER, FOR WHICH IRC ≈ 3 AND $t'(80) \approx 4$).