MAT 320 – Fall 2015 –Exam2

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I certify that I did not receive third party help in *completing* this test (sign) ______________________

Instructions. Technology is allowed on this exam. Each problem is worth 10 points. If you use formulas or properties from our book, make a reference. When using technology describe which commands (or keys typed) you used or print out and attach your worksheet.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1) Let $A = \begin{bmatrix} -3 & 2 \\ -1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$. Find $3A - 2B$.

$$3A-2B = \begin{bmatrix} -9 & 6 \\ -3 & 21 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} -9 & 6 \\ -3 & 21 \end{bmatrix} + \begin{bmatrix} -2 & +4 \\ 0 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 10 \\ -3 & 11 \end{bmatrix}$$

2) Find the transpose of $M = \begin{bmatrix} 2 & -7 \\ 1 & 5 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}$.

$$M^T = \begin{bmatrix} 2 & 1 & 3 & 4 \\ -7 & 5 & 6 & 8 \end{bmatrix}$$

$\quad\quad\quad\quad\quad B_1 \quad\quad\quad B_2 \quad\quad B_3$

3) Consider $\mathfrak{B} = \left\{\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\ 2 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}\right\}$, $A = \begin{bmatrix}0 & 2\\ 2 & -1\end{bmatrix}$, and $C = \begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}$.

a) Determine if A or C are in $Span(\mathfrak{B})$.

b) If a matrix in part a) is in the span, write it as a linear combination of the matrices in $\mathfrak{B}$.

a) $M \in Span\, \mathfrak{B}$ IF AND ONLY IF $t_1 B_1 + t_2 B_2 + t_3 B_3 = M$

OR $\begin{bmatrix} t_1 & -t_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} t_2 & t_2 \\ 2t_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ t_3 & -t_3 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$, THIS

CORRESPONDS TO A SYSTEM OF EQUATIONS IN t_1, t_2, t_3 WITH KNOWN TERMS $m_{11}, m_{21}, m_{12}, m_{22}$. WRITTEN BY COLUMNS, THE AUGMENTED MATRIX IS:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & m_{11} \\ 0 & 2 & 1 & m_{21} \\ -1 & 1 & 0 & m_{12} \\ 0 & 0 & -1 & m_{22} \end{array}\right]$$

$$A \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 2 \\ -1 & 1 & 0 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

INCONSISTENT, THEN $A \notin Span\, \mathfrak{B}$

$$C \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

CONSISTENT, THEN $C \in Span\, \mathfrak{B}$

b) FROM (a) $(t_1, t_2, t_3) = (1, 1, -1)$ THEN

$C = B_1 + B_2 - B_3$

4) Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 1 & 2 \end{bmatrix}$ and f_A be the corresponding matrix mapping. Determine the domain, the codomain, and the expression of f_A.

$f_A(\vec{x}) = A\vec{x}$, THEN $f_A : \mathbb{R}^2 \to \mathbb{R}^3$: $\boxed{\text{DOMAIN } \mathbb{R}^2}$, $\boxed{\text{CODOMAIN } \mathbb{R}^3}$

$$f_A(x_1, x_2) = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ -x_1 \\ x_1 + 2x_2 \end{bmatrix} = (2x_1 + 3x_2, -x_1, x_1 + 2x_2)$$

5) Suppose that T is a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^4$ such that $T(\boldsymbol{e}_1) = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$, $T(\boldsymbol{e}_2) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, and $T(\boldsymbol{e}_3) = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. Find a formula for the image of an arbitrary vector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$.

$T(\vec{x}) = [T]\vec{x}$ WHERE $[T] = [T(\vec{e}_1) \;\; T(\vec{e}_2) \;\; T(\vec{e}_3)]$

$$= \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \text{ THEN } T(\vec{x}) = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + 2x_3 \\ 2x_1 + x_3 \\ -x_2 \\ x_2 - x_3 \end{bmatrix}$$

OR $T(x_1, x_2, x_3) = (x_2 + 2x_3, 2x_1 + x_3, -x_2, x_2 - x_3)$

6) Suppose that S and T are linear mappings with matrices $[S]=\begin{bmatrix}0&1\\-1&1\\1&0\\2&1\end{bmatrix}$ and $[T]=\begin{bmatrix}-1&0\\3&1\end{bmatrix}$.

(a) Determine the domain and the codomain of S, and T.
(b) When defined, determine the matrix that represents $S\circ T$ and the one that represents $T\circ S$.
(c) Use the expressions of S, and T and composition of functions in order to check your results in (b).

a) $[S]$ IS 4×2 THEN $S:\mathbb{R}^2\to\mathbb{R}^4$
$[T]$ IS 2×2 THEN $T:\mathbb{R}^2\to\mathbb{R}^2$

b) $\underset{4\times 2}{[S\circ T]}=\underset{4\times 2}{[S]}\,\underset{2\times 2}{[T]}=\begin{bmatrix}3&1\\4&1\\-1&0\\1&1\end{bmatrix}$ AND $S\circ T:\mathbb{R}^2\to\mathbb{R}^4$

$\underset{2\times 2}{[T]}\,\underset{4\times 2}{[S]}$ IS NOT DEFINED, THEN $T\circ S$ IS NOT DEFINED

c) FROM (b): $(S\circ T)(x_1,x_2)=\begin{bmatrix}3&1\\4&1\\-1&0\\1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}3x_1+x_2\\4x_1+x_2\\-x_1\\x_1+x_2\end{bmatrix}$

$=(3x_1+x_2,\ 4x_1+x_2,\ -x_1,\ x_1+x_2)$

$T(x_1,x_2)=\begin{bmatrix}-1&0\\3&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-x_1\\3x_1+x_2\end{bmatrix}=(-x_1,\ 3x_1+x_2)$

$S(y_1,y_2)=\begin{bmatrix}0&1\\-1&1\\1&0\\2&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}y_2\\-y_1+y_2\\y_1\\2y_1+y_2\end{bmatrix}=(y_2,\ -y_1+y_2,\ y_1,\ 2y_1+y_2)$

$(S\circ T)(x_1,x_2)=S(T(x_1,x_2))=(3x_1+x_2,\ -(-x_1)+(3x_1+x_2),$
$-x_1,\ 2(-x_1)+(3x_1+x_2))=(3x_1+x_2,\ 4x_1+x_2,\ -x_1,\ x_1+x_2)$

7) Let $A = \begin{bmatrix} 0 & 1 & -1 & 3 \\ 1 & 2 & 0 & 1 \\ -1 & 3 & 4 & -1 \\ 2 & 0 & 2 & 1 \\ 0 & -1 & 2 & 0 \end{bmatrix}$ and f_A be the corresponding linear mapping. Determine the **domain**, the **codomain**, and a **basis** for $Ker(f_A)$, the **kernel** of f_A (or **nullspace** of A).

A IS 5×4 THEN $f_A : \mathbb{R}^4 \to \mathbb{R}^5$ (DOMAIN $\mathbb{R}^4$, CODOMAIN $\mathbb{R}^5$)

$RREF(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $Ker\, f_A = NULL(A)$

$\to$ ALL PIVOTS: THERE IS NONE FREE VARIABLE $\to$

$\to$ THERE IS ONLY THE TRIVIAL SOLUTION: $Ker\, f_A = \{\vec{0}\}$

BY DEFINITION THE BASIS OF $\{\vec{0}\}$ IS THE EMPTY SET $\emptyset$.

8) Let $B = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and f_B be the corresponding matrix mapping. Determine the domain, the codomain, and a basis for Range(f_B), the **range** of f_B (or Col(B), the **columnspace** of B).

B IS 4×3 THEN $f_B : \mathbb{R}^3 \to \mathbb{R}^4$.

TO FIND LIN. INDEP. COLUMNS WE LOOK FOR THE PIVOTAL ONES:

$RREF(B) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ THE FIRST TWO COLUMNS OF B ARE LINEARLY INDEPENDENT:

PIVOTS

$Col(B) = Span \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\}$

9) Let $A = \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 3 & 0 & 1 & -1 & 0 \\ -1 & 1 & 3 & 2 & -1 \end{bmatrix}$ and f_A be the corresponding linear mapping.

(a) Determine the domain and the codomain of f_A.

(b) Determine a basis for Col(A) and one for Null(A).

(c) Use part (b) to verify the Rank Theorem.

c) 3 VECTORS

a) A is 3×5 THEN $f_A : \mathbb{R}^5 \to \mathbb{R}^3$.

b) $\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & -7/15 & 0 \\ 0 & 1 & 0 & 1/3 & -1 \\ 0 & 0 & 1 & 2/5 & 0 \end{bmatrix}$ $\quad \mathrm{COL}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$

THEN $\mathrm{COL}(A) = \mathbb{R}^3$

PIVOTS $\to$ COL(A)

FREE $\to$ NULL(A) $= \begin{cases} t_1 = 7/15\, t_4 \\ t_2 = -\frac{1}{3} t_4 + t_5 \\ t_3 = -2/5\, t_4 \end{cases}$

$$\mathrm{NULL}(A) = \left\{ \begin{bmatrix} 7/15\, t_4 \\ -1/3\, t_4 + t_5 \\ -2/5\, t_4 \\ t_4 \\ t_5 \end{bmatrix} \in \mathbb{R}^5 \mid t_4, t_5 \in \mathbb{R} \right\} = \mathrm{Span}\left\{ \underbrace{\begin{bmatrix} 7/15 \\ -1/3 \\ -2/5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{2 VECTORS}} \right\}$$

c)

c) $3+2=5$

10) Use Row Reduction to verify that $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ is invertible and use A^{-1} to solve the system $Ax = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \begin{matrix} R_1 - R_3 \\ R_2 - R_3 \end{matrix}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \begin{matrix} R_1 + \frac{1}{2} R_2 \\ \frac{1}{2} R_2 \end{matrix} \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & -3/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] = [I \mid A^{-1}]$$

$$x = Ix = (A^{-1}A)x = A^{-1}(Ax) \overset{*}{=} \frac{1}{2}\begin{bmatrix} -1 & 1 & -3 \\ -1 & 1 & -1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -11 \\ -5 \\ 8 \end{bmatrix} = \begin{bmatrix} -11/2 \\ -5/2 \\ 4 \end{bmatrix}$$

$$*\text{NOTE: } A^{-1} = \begin{bmatrix} -1/2 & 1/2 & -3/2 \\ -1/2 & 1/2 & -1/2 \\ 1 & 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 & -3 \\ -1 & 1 & -1 \\ 2 & 0 & 2 \end{bmatrix}$$