Math 221- Spring 2010 - Test 2

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Instructions. Only calculators are allowed on this examination. Point values of each problem are indicated. **Always use the appropriate wording and units of measure in your answers (when applicable). SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).**

1. Compute the following limits (when defined). You **can not** justify your answer with a graph.

(a) (*20 pts*) $\lim_{x\to\frac{1}{2}^-} \frac{2x}{\cos \pi x} = +\infty$ $\frac{k}{0}$

PLUG TEST $f(.4999) = \frac{2 \cdot .4999}{\cos(\pi \cdot .4999)} > 0$

(b) (*20 pts*) $\lim_{x\to 16} \frac{4-\sqrt{x}}{x-16} = \frac{0}{0}$ SIMPLIFY

PLUG $x=16$

$$\frac{4-\sqrt{x}}{x-16} = \frac{4-\sqrt{x}}{x-16} \cdot \frac{4+\sqrt{x}}{4+\sqrt{x}} = \frac{16-x}{-(16-x)(4+\sqrt{x})} = \frac{-1}{4+\sqrt{x}}$$

$x \neq 16$

BY THE REPLACEMENT THEOREM:

$$\lim_{x\to 16} \frac{4-\sqrt{x}}{x-16} = \lim_{x\to 16} \frac{-1}{4+\sqrt{x}} = \frac{-1}{4+\sqrt{16}} = -\frac{1}{8}$$

A RATIONAL FUNCTION IS CONTINUOUS

(c) (*20 pts*) $\lim_{x\to 0} \frac{2\tan^2 x}{x} = 0/0$ SIMPLIFY, USING KNOWN LIMITS

PLUG $x=0$

$$\frac{2\tan^2 x}{x} = 2\frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{x} = 2\frac{\sin^2 x}{x^2}\frac{x^2}{\cos^2 x}\frac{1}{x}$$

$$= 2\left(\frac{\sin x}{x}\right)^2\cdot\frac{x}{\cos^2 x}.$$

$$\lim_{x\to 0} 2\frac{\tan^2 x}{x} = 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\lim_{x\to 0}\frac{x}{\cos^2 x} =$$

$$= 2\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2\cdot\frac{0}{1}$$

$$= 2\cdot 1^2\cdot 0 = 0$$

(d) (*20 pts*) For $f(x) = \begin{cases} \frac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \geq 0 \end{cases}$, $\lim_{x\to 0} f(x) =$

SINCE $f(x)$ IS PIECE-WISE DEFINED, WE MUST CHECK THE SIDED-LIMITS

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{\cos x - 1}{x} = 0$$ (KNOWN LIMIT)

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 5x = 5\cdot 0 = 0$$ (CONTINUOUS FUNCTION)

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0$$ THEREFORE THE LIMIT IS

DEFINED AND $\lim_{x\to 0} f(x) = 0$

(e) (*25 pts*) $\lim_{x\to 4}(3x-7) =$ $3\cdot 4 - 7 = 5$ (CONTINUOUS FUNCTION)
In this case, prove your result using the ϵ–δ definition. $\lim_{x\to c} f(x) = L$

WE MUST PROVE THAT FOR ANY $\varepsilon > 0$ THERE EXISTS $\delta > 0$ SUCH THAT IF $0 < |x - c| < \delta$ THEN $|f(x) - L| < \varepsilon$. ASSUME $x \neq c$.

HERE $|x-4| < \delta$ AND $\varepsilon > |(3x-7)-5| = |3x-12| = 3|x-4|$

WHEN $|x-4| < \delta$ THEN $3|x-4| < 3\delta$. WE JUST NEED TO SET $3|x-4| < 3\delta < \varepsilon$, SO THAT $3\delta < \varepsilon$ OR $\delta < \frac{\varepsilon}{3}$ IS WHAT WE CAN USE.

FOR INSTANCE, TAKE $\varepsilon = .06$, WE CAN COMPUTE, FOR $\delta = .01 < .02 = \frac{.06}{3}$,

$f(4+.06) - 5 = 3\cdot(4.01) - 7 - 5 = .03 < \varepsilon = .06$

2. (*20 pts*) Discuss the continuity of the function $f(x) = \begin{cases} \dfrac{\sin(1-x)}{1-x}, & x > 1 \\ 2x^3 - 1, & x \leq 1 \end{cases}$

I) FOR $x > 1$, $f(x)$ IS A QUOTIENT OF CONTINUOUS FUNCTIONS WITH NON-ZERO DENOMINATOR, THUS IT IS CONTINUOUS.

II) FOR $x < 1$, $f(x)$ IS A POLYNOMIAL, THUS IT IS CONTINUOUS

III) CONTINUITY AT $x = 1$.

1) $f(1) = 2\cdot 1^3 - 1 = 1$ IS REAL ✓

2) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x^3 - 1 = 2\cdot 1^3 - 1 = 1$

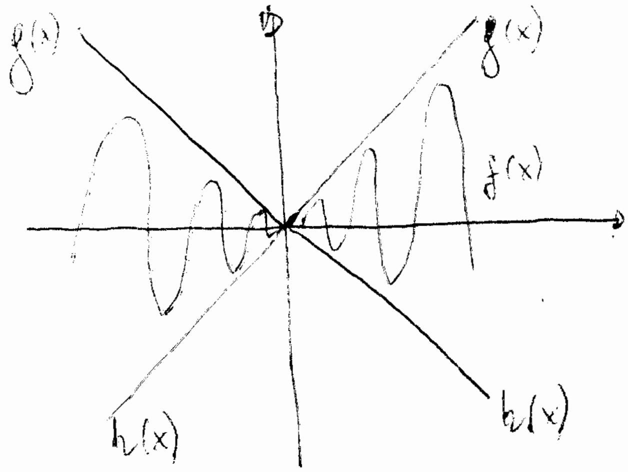
$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{\sin(1-x)}{1-x} = \lim_{y\to 0^-} \frac{\sin y}{y} = 1$ .. → SAME

THUS $\lim_{x\to 1} f(x) = 1$ ✓

3) $f(1) = \lim_{x\to 1} f(x)$ ✓

IV) $f(x)$ IS CONTINUOUS ON $(-\infty, +\infty)$

3. (*25 pts*) Graph the functions $h(x) = -|x|$, $f(x) = x \sin \frac{1}{x}$, and $g(x) = |x|$ in the same window of your graphing calculator. Use the Squeeze Theorem to find the limit $\lim_{x\to 0} f(x)$.



WE SEE FROM THE GRAPH THAT

I) $h(x) \le f(x) \le g(x)$.

MOREOVER:

II) $\lim_{x\to 0} h(x) = \lim_{x\to 0} g(x) = 0$

BY THE SQUEEZE THEOREM, USING I) AND II), WE HAVE:

$$\lim_{x\to 0} x \sin \frac{1}{x} = 0.$$