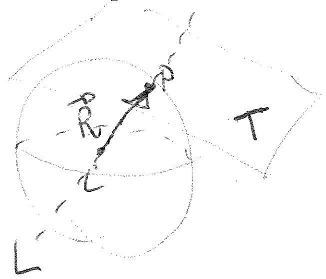
MAT 421 - Exam1 – Fall 2016 – Take Home Part

Instructor: Dr. Francesco Strazzullo **Name** KEY

Instructions. Complete 8 (9 for the Honor section) out of the following 10 exercises, according to the instructions. Each exercise is worth 10 points. When approximating, use four decimal places.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

Complete the following exercises.

1. Find an equation of the sphere with center $C = (-2,3,1)$ that is tangent to the plane $x - 2y + 3z = 1$.



$\vec{R} = \vec{CP}$ IS A DIRECTIONAL VECTOR OF T: $x-2y+3z=1$

$\vec{R} = t\langle 1,-2,3\rangle$ GENERATES THE LINE L

$L \equiv \begin{cases} x=-2+t \\ y=3-2t \\ z=1+3t \end{cases}$ WHICH CROSSES T ONLY AT P:

$P = L \cap T$: $(-2+t)-2(3-2t)+3(1+3t)=1 \Rightarrow -2+t-6+4t+3+9t=1 \Rightarrow$
$\Rightarrow 14t-5=1 \Rightarrow t=\frac{6}{14}=\frac{3}{7} \Rightarrow P \equiv \begin{cases} x=-2+3/7=-11/7 \\ y=3-2(3/7)=15/7 \\ z=1+3(3/7)=16/7 \end{cases} \Rightarrow R^2=\|\vec{CP}\|^2=$
$=(-2+11/7)^2+(3-15/7)^2+(1-16/7)^2=\frac{1}{49}\left((-3)^2+6^2+(-9)^2\right)=\frac{3^2}{49}(1+4+9)=$
$=\frac{9}{49}\cdot 14=\frac{18}{7}$. THEN $(x+2)^2+(y-3)^2+(z-1)^2=\frac{18}{7}$

2. Determine all values for a such that the vectors $\mathbf{x} = a\mathbf{i} - a\mathbf{j} + 3\mathbf{k}$ and $\mathbf{y} = 5a\mathbf{i} + a\mathbf{j} - \mathbf{k}$ are orthogonal.

ORTHOGONAL $\Leftrightarrow$ $\vec{x}\cdot\vec{y}=0 \Leftrightarrow 5a^2-a^2-3=0 \Leftrightarrow$

$\Leftrightarrow 4a^2=3 \Leftrightarrow a=\pm\sqrt{\frac{3}{4}}=\pm\frac{1}{2}\sqrt{3}$

3. Let $f(x, y) = \ln(x^2 - y^2 - 1) + 3x$.

(a) Evaluate $f(2,1)$.

(b) Find the domain of f (algebraically) and describe it graphically.

(c) Find the range of f.

(a) $f(2,1) = \ln(4-1-1) + 3(2) = 6 + \ln 2 \approx 6.6931$

(b) DOMAIN: $x^2 - y^2 - 1 > 0$

B.L. $x^2 - y^2 = 1$

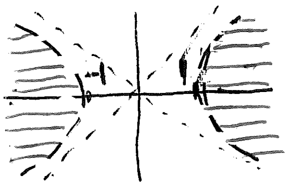
$(x-y)(x+y) = 1$

$x - y = t$

$x + y = \frac{1}{t}$

$x + y = \frac{1}{x-y}$ HYPERBOLE

TEST: $0 - 0 - 1 \not> 0$



(c) $\ln(w)$, $3x$ ARE ALL REAL:

RANGE $= \mathbb{R}$.

4. Let **a**(t), **v**(t), and **r**(t) denote the acceleration, velocity, and position at time t of an object moving in the *Euclidean Space*. Find **r**(t), given that

$$\mathbf{a}(t) = \langle t - \sin(\pi t), t^3, \ln t\rangle, \mathbf{v}(1) = \langle 1, 1, 0\rangle, \text{and } \mathbf{r}(1) = \langle 0, 0, 1\rangle.$$

$$\vec{v}(t) = \vec{v}(1) + \int_1^t \vec{a}(u)\,du = \langle 1,1,0\rangle + \left[\left\langle \frac{u^2}{2} + \frac{1}{\pi}\cos(\pi u), \frac{u^4}{4}, u(\ln u - 1)\right\rangle\right]_1^t$$

$$= \langle 1,1,0\rangle + \left\langle \frac{t^2}{2} + \frac{1}{\pi}\cos(\pi t), \frac{t^4}{4}, t(\ln t - 1)\right\rangle - \left\langle \frac{1}{2} - \frac{1}{\pi}, \frac{1}{4}, -1\right\rangle$$

$$= \left\langle \frac{t^2}{2} + \frac{1}{\pi}\cos(\pi t) + \frac{1}{2} + \frac{1}{\pi}, \frac{t^4}{4} + \frac{3}{4}, t\ln t - t + 1\right\rangle$$

$$\vec{r}(t) = \vec{r}(1) + \int_1^t \vec{v}(t) = \langle 0,0,1\rangle + \left[\left\langle \frac{t^3}{6} + \frac{1}{\pi^2}\sin(\pi t) + \frac{\pi+2}{2\pi}t, \frac{t^5}{20} + \frac{3}{4}t, \right.\right.$$

$$\left.\left. + \frac{1}{2}t^2\ln t - \frac{1}{4}t^2 - \frac{t^2}{2} + t\right\rangle - \left\langle \frac{1}{6} + \frac{1}{2} + \frac{1}{\pi}, \frac{1}{20} + \frac{3}{4}, -\frac{1}{4} - \frac{1}{2} + 1\right\rangle\right]$$

$$= \left\langle \frac{t^3}{6} + \frac{\pi+2}{2\pi}t + \frac{1}{\pi^2}\sin(\pi t) - \frac{2}{3} + \frac{1}{\pi}, \frac{t^5}{20} + \frac{3}{4}t - \frac{4}{5}, \frac{t^2}{4}(2\ln t - 3) + \right.$$

$$\left. + t + \frac{3}{4}\right\rangle$$

5. Write in cylindrical coordinates the parametric equations of the curve with vector equation

$$\mathbf{r}(t) = \langle t, -5\sin(2t), -5\cos(2t)\rangle.$$

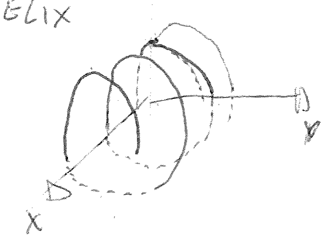
$$\begin{cases} X = r\cos\theta \\ Y = r\sin\theta \\ Z = Z \end{cases} \Rightarrow \vec{r}(t) \equiv \begin{cases} r^2 = t^2 + 25\sin^2(2t) \Rightarrow r = \sqrt{t^2+25\sin^2(2t)} \\ \theta = \arctan\left(-\dfrac{5\sin(2t)}{t}\right) \\ z = -5\cos(2t) \end{cases}$$

OR

$$\begin{cases} x^2+y^2 = r^2 \\ \tan\theta = \dfrac{y}{x} \\ z = z \end{cases}$$

ZD

HELIX



$$\vec{r}(t) \equiv \begin{cases} r = \sqrt{t^2+25\sin^2(2t)} \\ \theta = \arctan\left(-\dfrac{5\sin(2t)}{t}\right) \\ z = -5\cos(2t) \end{cases}$$

Complete two of the following three exercises.

6. Let $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$. Find an equation of the line parallel to $\mathbf{a} - \mathbf{b}$ and passing through the tip of $\mathbf{b}$.

$$\left.\begin{array}{l} \vec{a} - \vec{b} = \langle 1, -2, -5\rangle \text{ DIR. VECT.} \\ \text{THROUGH } (1, 1, -4) \end{array}\right\} \Rightarrow L \equiv \begin{cases} x = t+1 \\ y = -2t+1 \\ z = -5t-4 \end{cases}$$

7. Find parametric equations and symmetric equations of the line passing through the points $A = (0,1,-1)$ and $B = (1,-2,2)$.

DIRECTIONAL VECTOR $= B-A = \langle 1, -3, 3 \rangle$ THROUGH A $\Rightarrow L \equiv \begin{cases} x = t \\ y = -3t+1 \\ z = 3t-1 \end{cases}$

SYMMETRIC: $x = \dfrac{y-1}{-3} = \dfrac{z+1}{3}$ $\quad \left(\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c} \right)$

PAR: $\begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases}$

8. Let *L* be the line given by $x = 1+2t$, $y = 1 - 4t$, and $z = 3 + t$, and *P* be the point (3, 1, 1). Find parametric equations for the line through *P* which is parallel to *L*.

"DIRECT. VECTOR OF L" $= \langle 2, -4, 1 \rangle$ THROUGH P $\Rightarrow L \equiv \begin{cases} x = 2t+3 \\ y = -4t+1 \\ z = t+1 \end{cases}$

Complete one of the following two exercises (Honor section will complete both).

9. Find a point where the curve $\mathbf{r}(t) = \langle 3t, \frac{1}{3}t^3, 4t-1\rangle$ has maximum curvature. What is the maximum curvature?

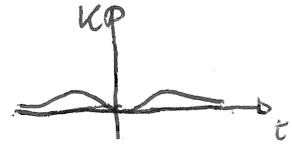
$$\vec{r}\,'(t) = \langle 3, t^2, 4\rangle \Rightarrow \|\vec{r}\,'\| = \sqrt{3^2+t^4+4^2} = \sqrt{25+t^4} \Rightarrow$$

$$\Rightarrow \vec{T}(t) = \frac{1}{\sqrt{25+t^4}}\langle 3, t^2, 4\rangle \Rightarrow \vec{T}\,'(t) = \frac{1}{\sqrt{25+t^4}}\langle 0, 2t, 0\rangle +$$

$$-\frac{2t^3}{(25+t^4)^{3/2}}\langle 3, t^2, 4\rangle = \frac{2t}{(25+t^4)^{3/2}}\langle -3t^2, t^4+25-t^4, -4t^2\rangle$$

$$\kappa(t) = \frac{\|T'\|}{\|r'\|} = \frac{|2t|(25+t^4)^{-3/2}}{(t^4+25)^{\frac{1}{2}}}\cdot\sqrt{9t^4+(25)^2+16t^4} \rightarrow \sqrt{25(t^4+25)}$$

$$\kappa(t) = \frac{10|t|}{(25+t^4)^{3/2}}$$



$\xrightarrow{\text{CAS}}$ MAX AT $t = \pm\sqrt[4]{5} \approx \pm 1.4953$

NOTE: $\kappa(t) = \frac{\|r'\times r''\|}{\|r'\|^3}$ IS BETTER IN THIS CASE

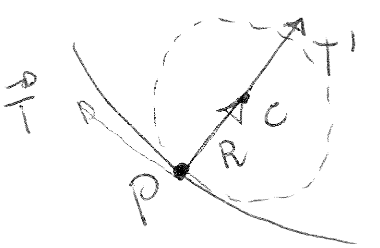
MAX CURVATURE $\kappa = \frac{\sqrt[4]{4500}}{90} \approx .091$

10. Find the center of the osculating circle of the curve $\mathbf{r}(t) = \langle 3t, \frac{1}{3}t^3, 4t-1\rangle$ at $P = \left(3, \frac{1}{3}, 3\right)$.

AT $P = \left(3, \frac{1}{3}, 3\right) = \vec{r}(t) \Rightarrow t = 1 \Rightarrow \kappa(1) = \frac{5\sqrt{26}}{26^2}$

$$R = \frac{1}{\kappa(1)} = \frac{13}{5}\sqrt{26}$$

$$\vec{T}\,'(1) = \frac{1}{13\sqrt{26}}\langle -3, 25, -4\rangle \Rightarrow \|\vec{T}\,'\| = \frac{5\cancel{\sqrt{26}}}{13\cancel{\sqrt{26}}}$$



$$\vec{N}(1) = \frac{\vec{T}\,'}{\|T'\|} = \frac{1}{5\sqrt{26}}\langle -3, 25, -4\rangle \Rightarrow$$

$$\Rightarrow C - P = R\cdot\vec{N}(1) = \frac{13}{25}\langle -3, 25, -4\rangle \Rightarrow$$

$$\Rightarrow C = \left(\frac{(-3)(13)}{25}+3,\ 13+\frac{1}{3},\ \frac{(-4)(13)}{25}+3\right) =$$

$$= \left(\frac{12}{25}, \frac{40}{3}, \frac{23}{25}\right)$$

MAT 421 - Exam1 – Fall 2016 – In Class Part

Instructor: Dr. Francesco Strazzullo **Name** KEY

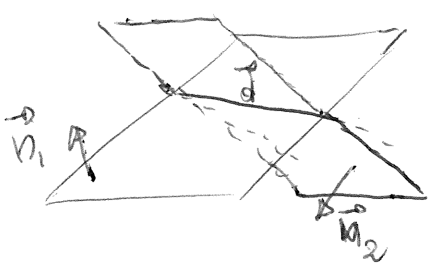
Instructions. Complete 2 out of the following 5 exercises, according to the instructions. Each exercise is worth 10 points.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

Complete one of the following two exercises.
1. Find parametric equations of the tangent line to the curve $\mathbf{r}(t) = \langle \ln t, t^2, 2\ln t\rangle$ at $P = (1, 1, 2)$.

$$\left.\begin{array}{l} \vec{r}\,'(t) = \left\langle \frac{1}{t}, 2t, \frac{2}{t} \right\rangle \\ \vec{r}(t) = \langle P \rangle = \langle 1, 1, 2 \rangle \Rightarrow t = 1 \end{array}\right\} \Rightarrow \text{DIR. VECT} = \vec{r}\,'(1) = \langle 1, 2, 2 \rangle$$

$$L \equiv \begin{cases} x = t + 1 \\ y = 2t + 1 \\ z = 2t + 2 \end{cases}$$

2. Find the directional vector of the line which is the intersection of the planes $x + 2y - 3z = 1$ and $2x - y + 2z = -1$.



LINE PARALLEL TO BOTH PLANES, THEN PERPENDICULAR TO BOTH DIRECTIONAL VECTORS.

$$\vec{d} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -3 \\ 2 & -1 & 2 \end{vmatrix} =$$

$$= \left\langle \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix}, -\begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \right\rangle$$

$$= \langle 4 - 3, -(2 + 6), -1 - 4 \rangle = \langle 1, -8, -5 \rangle$$

Complete one of the following three exercises.

3. Find the unit tangent and the unit normal to the trajectory of the vector function $\mathbf{r}(t) = \langle 2t, t^3, 0 \rangle$ at $P = (2,1,0)$.

$\vec{r}(t) = \langle P \rangle \Rightarrow t = 1.$; $\vec{r}'(t) = \langle 2, 3t^2, 0 \rangle$

UNIT TANGENT: $\vec{T}(t) = \frac{1}{\sqrt{4+9t^4}} \langle 2, 3t^2, 0 \rangle$

UNIT NORMAL: $\vec{N}(1) = \left.\frac{\vec{T}'}{\|\vec{T}'\|}\right|_{t=1}$

$$\vec{T}' = \frac{-18t^3}{(4+9t^4)^{3/2}} \langle 2, 3t^2, 0 \rangle + (4+9t^4)^{-\frac{1}{2}} \langle 0, 6t, 0 \rangle \Rightarrow$$

$$\Rightarrow \vec{T}'(1) = (13)^{-3/2}(-18)\langle 2, 3, 0 \rangle + (13)^{-\frac{1}{2}} \langle 0, 6, 0 \rangle$$

$$= (13)^{-3/2} \langle -36, -54 + 13 \cdot 6, 0 \rangle = \frac{1}{13\sqrt{13}} \langle -36, 24, 0 \rangle$$

$$= \frac{12}{13\sqrt{13}} \langle -3, 2, 0 \rangle$$

$$\|\vec{T}'(1)\| = \frac{12}{13}.$$

THEN AT P:

$$\vec{T}(1) = \frac{1}{\sqrt{13}} \langle 2, 3, 0 \rangle$$

$$\vec{N}(1) = \frac{1}{\sqrt{13}} \langle -3, 2, 0 \rangle$$

4. Let **a**(t), **v**(t), and **r**(t) denote the acceleration, velocity, and position at time t of an object moving in the space. Find **r**(t), given that

$$\mathbf{a}(t) = \langle \sin 2t, t, 1 \rangle, \mathbf{v}(0) = \langle 1, -1, 1 \rangle, \text{and } \mathbf{r}(0) = \langle 2, 1, 2 \rangle.$$

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)\,du = <1,-1,1> + \left[<-\frac{1}{2}\cos u, \frac{u^2}{2}, u>\right]_0^t =$$

$$= <1,-1,1> + <-\frac{1}{2}\cos(2t), \frac{t^2}{2}, t> - <-\frac{1}{2}, 0, 0> =$$

$$= <-\frac{1}{2}\cos(2t) + \frac{3}{2}, \frac{t^2}{2} - 1, t+1>$$

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)\,du = <2,1,2> + \left[<-\frac{1}{4}\sin(2u) + \frac{3}{2}u, \frac{u^3}{6} - u, \frac{u^2}{2} + u\right]_0^t =$$

$$= <2,1,2> + <-\frac{1}{4}\sin(2t) + \frac{3}{2}t, \frac{t^3}{6} - t, \frac{t^2}{2} + t> - \vec{0}$$

$$= <-\frac{1}{4}\sin(2t) + \frac{3}{2}t + 2, \frac{t^3}{6} - t + 1, \frac{t^2}{2} + t + 2>$$

5. Find a parametric representation for the curve of intersection between the ellipsoid

$$\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1$$

and the plane $x = 1$.

$$\vec{r} \equiv \begin{cases} \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1 \\ x = 1 \end{cases} \equiv \begin{cases} \frac{y^2}{16} + \frac{z^2}{9} = \frac{24}{25} \\ x = 1 \end{cases}$$

$$\frac{y^2}{16} + \frac{z^2}{9} = \frac{24}{25} \Leftrightarrow \frac{y^2}{\frac{16 \cdot 25}{24}} + \frac{z^2}{\frac{9 \cdot 25}{24}} = 1 \Leftrightarrow \frac{y^2}{\frac{2}{3}5^2} + \frac{z^2}{\frac{3}{2}\left(\frac{5}{2}\right)^2} = 1 \Rightarrow$$

$$\Rightarrow \text{SET } y = 5\sqrt{\frac{2}{3}}\cos t : \left(\frac{z}{\sqrt{\frac{3}{2}}\left(\frac{5}{2}\right)}\right)^2 = 1 - \cos^2 t = \sin^2 t \Rightarrow z = \frac{5}{2}\sqrt{\frac{3}{2}}\sin t$$

$$\vec{r}(t) = \begin{cases} x = 1 \\ y = 5\sqrt{\frac{2}{3}}\cos t \\ z = \frac{5}{2}\sqrt{\frac{3}{2}}\sin t \end{cases}$$