Mat321 – Spring 2019 –Exam3-Take Home

Instructor: Dr. Francesco Strazzullo **Name**________________________________

I certify that I did not receive third party help in *completing* this test (sign)____________________________

Instructions. Technology and instructor's notes (including the formula sheets from our book) are allowed on this exam. Each problem is worth 10 points, unless otherwise specified. If you use notes or formula sheets, make a reference. Some exercises **require** the use of Geogebra and you **must upload** to Eagleweb the GGB file you create.

SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Suppose that you have a water tank that holds 250 gallons of water. A briny solution, which contains 30 grams of salt per gallon, enters the tank at the rate of 2 gallons per minute. At the same time, the solution is well mixed, and water is pumped out of the tank at the rate of 2 gallons per minute.
   (a) Since 2 gallons enters the tank every minute and 2 gallons leaves every minute, what can you conclude about the volume of water in the tank.
   (b) How many grams of salt enters the tank every minute?
   (c) Suppose that $S(t)$ denotes the number of grams of salt in the tank in minute t. How many grams are there in each gallon in minute t?
   (d) Since water leaves the tank at 2 gallons per minute, how many grams of salt leave the tank each minute?
   (e) Write a differential equation that expresses the total rate of change of S.

(a) V = VOLUME OF WATER = 250 GL IS CONSTANT

(b) LET $S(t)$ BE SALT IN gr AT TIME t MINUTES:

SALT IN AT A RATE OF "VOLUME IN" · "DENSITY OF SALT IN" (CONCENTRATION), THEN:

$$\text{RATE-IN} = 2\ \text{GL}/\text{MIN} \cdot 30\ \text{gr}/\text{GL} = 60\ \text{gr}/\text{MIN}$$

(c) $S(t) = V \cdot$ "CONCENTRATION AT TIME t" $\Rightarrow$ CONCENTRATION $= \dfrac{S(t)}{250}$ gr/GL

(d) $$\text{RATE-OUT} = 2\ \text{GL}/\text{MIN} \cdot \frac{S(t)}{250}\ \text{gr}/\text{GL} = \frac{S(t)}{125}\ \text{gr}/\text{MIN}$$

(e) $$\frac{dS}{dt} = \text{RATE-IN} - \text{RATE-OUT} = 60 - \frac{S(t)}{125}\ \text{gr}/\text{MIN}$$

2. Show that $f(x) = 2x + 2 - 7e^{x-2}$ is the solution of the IVP
$y' = y - 2x, \quad y(2) = -1.$

$f'(x) = 2 - 7e^{x-2} = LHS$

$RHS = f(x) - 2x = 2 - 7e^{x-2}$ ✓

NOTE: FIRST CHECK IV: $f(2) = 2(2) + 2 - 7e^{2-2} = 6 - 7 = -1$ ✓

3. Use Euler's method with step $h = \Delta x = 0.05$ to estimate $f(2.5)$ for the solution of the IVP
$y' = y - 2x, \quad y(2) = -1.$
Show your setup with formula, then use technology (a spreadsheet) to perform computations (upload it if you do not copy your computations here). Finally, use exercise **2** to compute the approximation error.

$X_0 = 2;\ Y_0 = -1;\quad X_n = X_0 + nh,\quad Y_{n+1} = Y_n + h\,F(X_n, Y_n) =$

$= Y_n + h(Y_n - 2X_n) = (h+1)Y_n - 2hX_n = (h+1)Y_n - 2hX_0 - 2nh^2$

THEN $X_n = .05n + 2$ AND $Y_n = 1.05\,Y_{n-1} - .2 - .005(n-1)$.

$(X_1, Y_1) = (2.05, -1.25);\ ---$

$(X_3, Y_3) = (2.15, -1.8034);\ ---$

$(X_5, Y_5) = (2.25, -2.434);\ ---$

$---;\ (X_8, Y_8) = (2.4, -3.5422)$

$---;\ (X_{10}, Y_{10}) = (2.5, -4.4023).$

THEREFORE: $Y_{10} = Y(2.5) \approx -4.4023.$

FROM #2: $f(2.5) \approx -4.541$

ERROR $= |f(2.5) - Y_{10}| \approx .1388$

4. Find the (explicit) solution of the IVP given by $y' = y\cos x - y$ and $y(0) = 2$.

$y' = y(\cos x - 1)$ EQ. SOL. $y = 0$, NOT SOL. OF IVP.

$\int \frac{1}{y}\,dy = \int \cos x - 1\,dx \Rightarrow \ln|y| = \sin x - x + C_1$] $\Rightarrow$

$y(0) = 2 > 0 \Rightarrow$ SOLUTION WILL BE $y > 0$

$\Rightarrow \ln 2 = 0 - 0 + C_1 \Rightarrow C_1 = \ln 2$. THEREFORE

$y = e^{\sin x - x + \ln 2} = 2e^{\sin x - x}$

5. Consider the differential equation $y' = (4y^2 - 4y - 3)x$. (HONOR use $y' = (y^3 - y^2 - y)x$)
(a) What are the equilibrium solutions?
(b) Without using graphs, find the values of y for which $y(t)$ is increasing or decreasing as $x \to \infty$.
(c) Using part (b), decide the stability of the equilibrium solutions of y.
(d) Use a software to sketch the direction field and the equilibrium solutions for the given differential equation and compare it with the information from above (**upload your file**).

(a) $Y' = (2Y+1)(2Y-3)X \Rightarrow$ EQ. SOL. $Y = -\frac{1}{2}$ OR $Y = \frac{3}{2}$

HONOR: $Y' = Y(Y^2 - Y - 1)X$
$Y^2 - Y - 1 = 0 \Rightarrow Y = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$ $\Bigg] \Rightarrow$ EQ. SOL: $Y = \frac{1-\sqrt{5}}{2}, 0, \frac{1+\sqrt{5}}{2}$

(b) ASSUMING $X > 0$:

	$-1/2$		$3/2$	
Y	-1	0	2
Y'	$+$	$-$	$+$

INCREASING FOR $Y < -\frac{1}{2}$ OR $Y > \frac{3}{2}$
DECREASING FOR $-\frac{1}{2} < Y < 3/2$

(c) $Y = -\frac{1}{2}$ STABLE AND $Y = \frac{3}{2}$ UNSTABLE

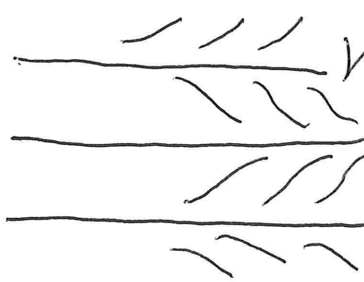
HONOR (b)

$X > 0$

	$\frac{1-\sqrt{5}}{2}$		0		$\frac{1+\sqrt{5}}{2}$	
Y	-3	$-\frac{1}{2}$	1	3
Y'	$-$	$+$	$-$	$+$

INCREASING FOR $\frac{1-\sqrt{5}}{2} < Y < 0$ OR $Y > \frac{1+\sqrt{5}}{2}$
DECREASING FOR $Y < \frac{1-\sqrt{5}}{2}$ OR $0 < Y < \frac{1+\sqrt{5}}{2}$

(c)



$Y = \frac{1+\sqrt{5}}{2}$ UNSTABLE
$Y = 0$ STABLE
$Y = \frac{1-\sqrt{5}}{2}$ UNSTABLE

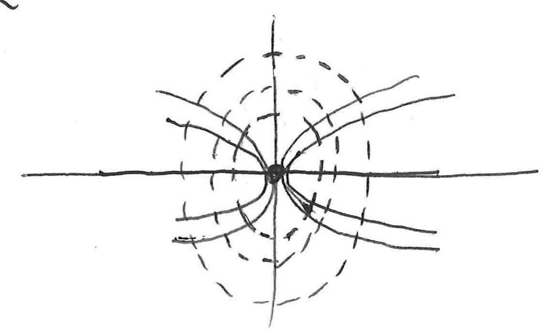
6. Find the orthogonal trajectories of the family of curves below, then draw several members of each family on the same coordinate plane and **upload a GGB file** with these families of functions.
$y^2 = kx$
HONOR use $y^2 = k\cos x$

$$y^2 = kx \Rightarrow 2yy' = k = \frac{y^2}{x} \Rightarrow y' = \frac{y}{2x} \Bigg] \Rightarrow$$

ORTHOGONAL TRAJS. SATISFY THE ODE $f'(x) = -\frac{1}{y'}$

SOLVE ODE: $y' = -\frac{2x}{y} \Rightarrow \int y\,dy = -2\int x\,dx \Rightarrow$

$$\Rightarrow \frac{y^2}{2} = -x^2 + C \Rightarrow x^2 + \frac{y^2}{2} = C \qquad \text{ELLIPSES}$$



— GIVEN FAMILY

---- ORTH. TRAJS.

NOTE: • K ANY REAL

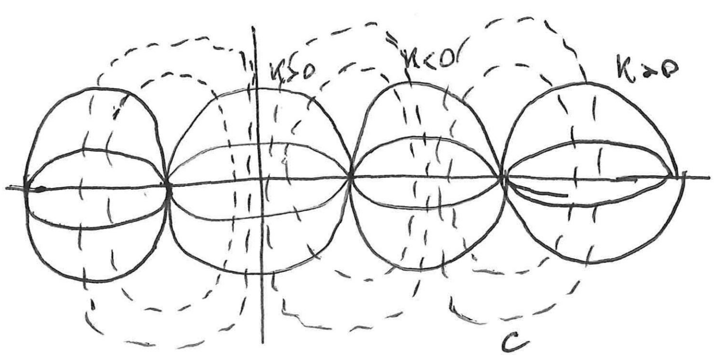
• $C \geq 0$

—

HONOR: $2yy' = -k\sin x = -\frac{y^2}{\cos x}\sin x \Rightarrow y' = -\frac{1}{2}y\tan x$

ORTH. TRAJS: $y' = +\frac{2\cot x}{y} \Rightarrow \int y\,dy = +2\int \cot x\,dx \Rightarrow$

$$\Rightarrow \frac{y^2}{2} = +2\ln|\sin x| + C \Rightarrow \frac{y^2}{2} - 2\ln|\sin x| = C$$



PERIODIC LOOPS

7. In a model of epidemics, the number of infected individuals in a population at a time is a solution of the logistic differential equation $\frac{dy}{dt} = .0032y^2 - .04y$, where y is the number of infected individuals in the community and t is the time in days.
(a) Find the most general expression that can describe the infected population.
(b) Assume that 18 people were infected at the initial time $t = 0$. Express the infected population.
(c) What is the limiting size for the number of infected individuals?
(d) How many days will it take for half of the population to be infected?

(a) $Y' = .0032Y\left(Y - \frac{25}{2}\right) \Rightarrow \int \frac{dY}{Y(Y-12.5)} = \int .0032\,dt \Rightarrow$

PARTIAL $\Rightarrow$ FRACTIONS $\quad A\left(Y-\frac{25}{2}\right) + BY = 1 \Rightarrow \left[\begin{array}{l} Y=\frac{25}{2} \to B = 2/25 \\ Y=0 \to A = -2/25 \end{array}\right.$ EQUIV. SOL. $\quad$ THEN

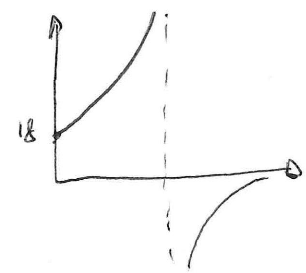
$\frac{25}{2}\left(\int \frac{-2/25}{Y} + \frac{2/25}{Y-\frac{25}{2}}\,dY = .0032t + c_1\right) \Rightarrow -\ln|Y| + \ln|Y-12.5| = \frac{1}{25}t + c_2 \Rightarrow$

$\Rightarrow \ln\left|\frac{Y-12.5}{Y}\right| = \frac{t}{25} + c_2 \Rightarrow \frac{Y-12.5}{Y} = ce^{.04t} \Rightarrow Y - 12.5 = ce^{.04t}Y$

$\Rightarrow Y = \frac{12.5}{1-ce^{.04t}} \Rightarrow$ GEN. SOL. $\left[\begin{array}{l} Y = 0 \\ \text{OR} \\ Y = \frac{12.5}{1-ce^{.04t}} \end{array}\right.$

(b) $18 = Y(0) = \frac{12.5}{1-ce^{0}} = \frac{12.5}{1-c} \Rightarrow 1 - c = \frac{12.5}{18} \Rightarrow c = \frac{11}{36} \Rightarrow$

$\Rightarrow Y = \frac{25/2}{1-\frac{11}{36}e^{.04t}} = \frac{450}{36-11e^{.04t}}$

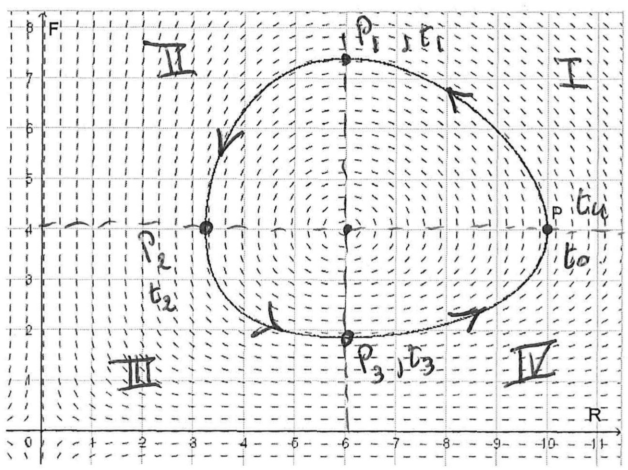


(c) LIMITING SIZE $= \lim_{t\to\infty} Y = 0$, BUT IN THIS CONTEXT THERE IS A V.A. AT DAY $t = 25\ln\left(\frac{36}{11}\right) \approx 30$

(d) THIS QUESTION CANNOT BE ANSWERED, BECAUSE THERE IS NONE LIMITING SIZE FOR THE POPULATION OF INFECTED INDIVIDUALS. IF WE HAD THE POPULATION'S SIZE, THEN WE COULD ANSWER.

* NOTE: THE LOGISTIC MODEL WAS THE OPPOSITE: $Y' = -.0032Y^2 + .04Y$

8. A phase portrait of a predator-prey system is given below in which F represents the population of foxes and R the population of rabbits (in tens).



(a) Referring to the graph, what is a reasonable non-zero equilibrium solution for the system?
(b) Write down a possible system of differential equations which could have been used to produce the given graph.
(c) Describe how each population changes as time passes, using the initial condition P indicated on the graph.
(d) Use your description in part (c) to make a rough sketch of the graph of R and F as functions of time.
(e) Recreate the phase portrait above in a GGB file and **upload your file.**

(a) $E = (6,4) = \left(\frac{r}{b}, \frac{k}{a}\right)$ FOR $\begin{cases} \frac{dR}{dt} = kR - aRF \\ \frac{dF}{dt} = -rF + bRF \end{cases}$

$r = 6b$; $k = 4a$

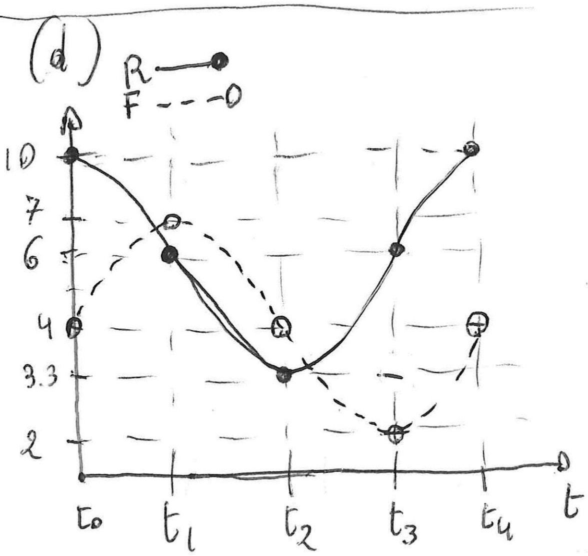
(b) CHOOSING $a = b = 1$:
(ANSWERS VARY)

$\begin{cases} \dot{R} = 4R - RF \\ \dot{F} = -6F + RF \end{cases}$ SO $\frac{dF}{dR} = \frac{(R-6)F}{(4-F)R}$

(c) WE HAVE FOUR "SEASONS" STARTING A TIME t_0 WHEN $(R,F) = P = (10,4)$ AND ENDING AT TIME t_4 WHEN WE HAVE AGAIN $(R,F) = (10,4)$.
NOTE: R IS TENS OF RABBITS, SO WE CAN TRUNCATE R TO 1-DECIMAL.

- $t_0 \le t \le t_1$ (FROM $(10,4)$ TO $(6,7)$)
  R DECREASING, F INCREASING
- $t_1 \le t \le t_2$ ($(6,7)$ TO $(3.3,4)$)
  R AND F DECREASING
- $t_2 \le t \le t_3$ ($(3.3,4)$ TO $(6,2)$)
  R INCREASING AND F DECREASING
- $t_3 \le t \le t_4$ ($(6,2)$ TO $(10,4)$)
  BOTH INCREASING

(d)



MAT 321 – Spring 2019 – Exam3 – In Class

Instructor: Dr. Francesco Strazzullo **Name**______________________________

I certify that I did not receive third party help in *completing* this test (sign) ______________________________

Instructions. Each exercise is worth 10 points. Same instructions as for the take-home part.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

9. Find the explicit general solution of $y' = yx - 2x$ and the particular solution given by $y(-1) = 1$.

$y' = x(y-2)$ EQ. SOL. $y = 2$. BY INTEGRATION

$\int \frac{1}{y-2}\,dy = \int x\,dx \Rightarrow \ln|y-2| = \frac{1}{2}x^2 + c_1 \Rightarrow |y-2| = e^{\frac{1}{2}x^2 + c_1}$

$c_2 = e^{c_1}$, $c_2 > 0$ $\Rightarrow y - 2 = \pm c_2\, e^{\frac{1}{2}x^2}$ $\quad c_3 = \pm c_2$, $c_3 \neq 0$ $\Rightarrow y = 2 + c_3\, e^{\frac{1}{2}x^2}$. IF $c = \left[\begin{array}{l} c_3 \\ \text{OR} \\ 0 \end{array}\right.$ THEN

- GEN. SOL. $y = 2 + c\, e^{\frac{1}{2}x^2}$ (EQUIL. SOL. FOR $c = 0$)

- PLUG I.V.: $1 = 2 + c\, e^{\frac{1}{2}(-1)^2} = 2 + c\sqrt{e} \Rightarrow c = -\frac{1}{\sqrt{e}} \Rightarrow$

$\Rightarrow$ SOL. IVP IS $y = 2 - e^{\frac{1}{2}(x^2-1)}$

10. Use Euler's method with step $h = \Delta x = 0.02$ to estimate $y(-1.4)$ for the IVP given by $y' = yx - 2x$ and $y(-1) = 1$. Show your setup formula, then use technology (a spreadsheet) to perform computations (upload it if you do not copy the results of your computations here). Finally, use exercise 9 to compute the approximation error.

STEP $h = .02$ FROM $X_0 = -1$ TO $X_n = -1.4$, WE NEED TO

MOVE LEFTWARD $\dfrac{|-1.4-(-1)|}{h} = 20$ THEN $Y(-1.4) \approx Y_{20}$.

$(X_0, Y_0) = (-1, 1)$, $X_n = X_{n-1} - h$, $Y_n = Y_{n-1} - h\,F(X_{n-1}, Y_{n-1}) \Rightarrow$

$\Rightarrow Y_n = Y_{n-1} - .02\,X_{n-1}(Y_{n-1} - 2)$.

$(X_1, Y_1) = (-1.02, .98)$; $(X_2, Y_2) = (-1.04, .9592)$; ...

$Y_5 = .8916$; ...; $Y_8 = .8154$; ...; $Y_{12} = .6986$;

...; $Y_{16} = .5612$; ...; $Y_{19} = .4424$; $\boxed{Y_{20} = .3994}$

$Y(-1.4) = y(-1.4) = 2 - e^{\frac{1}{2}((-1.4)^2 - 1)} \approx .3839$

ERROR $= |Y_{20} - Y(-1.4)| \approx .0155$