Mat321 – Spring 2015 –Exam4

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I certify that I did not receive third party help in *completing* this test (sign) ____________

Instructions. Technology and instructor's notes (including the formula sheets from our book) are allowed on this exam. Each problem is worth 10 points. **Complete 2 of the 10 exercises in class and mark them.** When using technology describe which one you used or print out your worksheet. You are receiving two copies of this booklet, one you take home and the other must be turned in with your two marked exercise. Please, submit the test online and return your completed take home copy by Monday (4/20).

SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Find the limit of the sequence $\left\{\sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, \dots\right\}$.

$a_1 = \sqrt{3}$, $a_2 = \sqrt{3\sqrt{3}} = \sqrt{3a_1}$, $a_3 = \sqrt{3a_2}$, ...

$a_{n+1} = \sqrt{3a_n}$. BY JUST LOOKING AT POINTS (n, a_n) WE CAN CONCLUDE THAT $\{a_n\}$ IS INCREASING AND BOUNDED (<4). THEN $a_n \to L$ AND BY RECURRENCE $L = \sqrt{3L} \Rightarrow$

$\Rightarrow L^2 = 3L \Rightarrow L(L-3) = 0 \Rightarrow L = 0$ OR $L = 3$, BUT $a_n > 0$ THEN $L = 3$ AND $a_n \to 3$

2. If $a_1 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$ for $n \geq 1$ and $\lim\limits_{n \to \infty} a_n = L$ is assumed to exist, then what must L be?

BY RECURSION: $L = \sqrt{1+L} \Rightarrow L^2 = 1 + L \Rightarrow L^2 - L - 1 = 0$

$\left.\begin{array}{l} \Rightarrow L = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2} \\ \{a_n\} \text{ IS INCREASING AND } a_1 \geq 1 \Rightarrow L \geq 1 \end{array}\right] \Rightarrow$

$\Rightarrow L = \dfrac{1 + \sqrt{5}}{2}$

3. A rubber ball is dropped from a height of 10 feet and bounces to $\frac{3}{4}$ its height after each fall. If it continues to bounce until it comes to rest, find the total distance in feet it travels.

BOUNCINGS	0	1	2	---	n
HEIGHT	10	$\frac{3}{4}(10)$	$\frac{3}{4}\left(\frac{3}{4}(10)\right)$	. . .	$\left(\frac{3}{4}\right)^n \cdot 10$

DISTANCE TRAVELLED DOWNWARD $= 10 + \frac{3}{4}(10) + \left(\frac{3}{4}\right)^2 10 + \dots + \left(\frac{3}{4}\right)^n \cdot 10 + \dots$

$$= \sum_{n=0}^{\infty} 10\left(\frac{3}{4}\right)^n = \frac{a}{1-r} = \frac{10}{1-\frac{3}{4}} = 40 \text{ FEET}$$

GEOMETRIC SERIES $a = 10$, $r = \frac{3}{4}$

DISTANCE TRAVELLED UPWARD $= \underbrace{\frac{3}{4}(10)}_{a} + \underbrace{10\left(\frac{3}{4}\right)^2}_{a \cdot r} + \dots + \left(\frac{3}{4}\right)^n 10 + \dots = \sum_{n=0}^{\infty} \frac{15}{2}\left(\frac{3}{4}\right)^n = \frac{\frac{15}{2}}{1-\frac{3}{4}} = 30$ FEET

TOTAL DISTANCE $= 40 + 30 = \boxed{70 \text{ FEET}}$

4. Find the sum of the series $\sum_{n=3}^{\infty} \frac{1}{4n^2-1}$. $= \sum_{n=3}^{\infty} \frac{1}{(2n+1)(2n-1)}$

$$\frac{1}{(2a+1)(2a-1)} = \frac{A}{2a+1} + \frac{B}{2a-1} = \frac{(2a-1)A + (2a+1)B}{4a^2-1} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} a = \frac{1}{2} \to 2B = 1 \Rightarrow B = 1/2 \\ a = -\frac{1}{2} \to -2A = 1 \Rightarrow A = -\frac{1}{2} \end{bmatrix} \Rightarrow \sum_{n=3}^{\infty} \frac{1}{4n^2-1} = \sum_{n=3}^{\infty} \frac{1/2}{(2n-1)} - \frac{1/2}{(2n+1)}$$

$$= \frac{1}{2}\sum_{n=3}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2}\Big(\frac{1}{5} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{7}} - \frac{1}{9} + \dots + \underbrace{\frac{1}{2n-1} - \cancel{\frac{1}{2n+1}}}_{n\text{-TH}} +$$

$$+ \underbrace{\cancel{\frac{1}{2(n+1)-1}} - \frac{1}{2(n+1)+1}}_{(n+1)\text{-TH}} + \dots\Big) = \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}$$

5. Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{(-3x)^n}{3n+1}$.

RATIO TEST: $a_n = \frac{(-3)^n x^n}{3n+1} \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{(-1)^{n+1} 3^{n+1} x^{n+1}}{3(n+1)+1}}{\frac{(-1)^n 3^n x^n}{3n+1}} \right| =$

$= 3 \frac{3n+1}{3n+4} \cdot |x| \xrightarrow[n \to \infty]{} 3|x| < 1 \Rightarrow |x| < \frac{1}{3} \Rightarrow$

$\Rightarrow -\frac{1}{3} < x < \frac{1}{3}$. CHECK END-POINTS:

- $x = \frac{1}{3} \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1}$ IS CONVERGENT BY ALT. SEQ. TH.

- $x = -\frac{1}{3} \Rightarrow \sum_{n=0}^{\infty} \frac{1}{3n+1} = 1 + \sum_{n=1}^{\infty} \frac{1}{3n+1} \geq 1 + \sum_{n=1}^{\infty} \frac{1}{3n+3} = 1 + \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n+1}$

$= 1 + \underbrace{\frac{1}{3}\left(-1\right.}_{-\frac{1}{3}} + \underset{n=1,\ 2,\ 3}{1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right)}\left.\right) = \frac{2}{3} + \underbrace{\sum_{k=1}^{\infty} \frac{1}{k}}_{\text{HARMONIC}} \longrightarrow \infty \Rightarrow$ DIV.

INTERVAL $\left[-\frac{1}{3}, \frac{1}{3}\right]$

6. Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{n}{4^n}(x+3)^n$.

AS ABOVE: $\left| a_{n+1} \cdot \frac{1}{a_n} \right| = \left| \frac{(n+1)}{4^{n+1}} \frac{(x+3)^{n+1} \cdot 4^n}{n(x+3)^n} \right| = \frac{1}{4} \frac{n+1}{n} |x+3| \longrightarrow \frac{1}{4}|x+3| \Rightarrow$

$\Rightarrow \frac{1}{4}|x+3| < 1 \Rightarrow |x+3| < 4 \Rightarrow$ INTERVAL $(\underset{-3-4}{-7}, \underset{-3+4}{1})$

END POINTS:

- $x = 1 \Rightarrow \sum_{n=0}^{\infty} n\left(\frac{4}{4}\right)^n = \sum_{n=1}^{\infty} n$ DIVERGENT
- $x = -7 \Rightarrow \sum_{n=0}^{\infty} n\left(\frac{-7+3}{4}\right)^n = \sum_{n=0}^{\infty} (-1)^n n$ DIVERGENT

INTERVAL $(-7, 1)$

7. Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(-3)^n (x-1)^n}{\sqrt{n+1}}$.

RATIO TEST: $\left| \frac{(-3)^{n+1}(x-1)^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n (x-1)^n} \right| = 3\sqrt{\frac{n+1}{n+2}}\,|x-1| \to 3|x-1| < 1 \Rightarrow$

$\Rightarrow |x-1| < \frac{1}{3} \Rightarrow$ CENTERED AT 1 WITH RADIUS $1/3$

8. Which of the following is the power series centered at $a = 0$ for $f(x) = \frac{x}{1+4x}$?

1) $\sum_{n=0}^{\infty} (-1)^n 4^n x^n$ 2) $\sum_{n=0}^{\infty} 4^n x^{n+1}$ $\boxed{3) \sum_{n=0}^{\infty} (-1)^n 4^n x^{n+1}}$

MACLAURIN SERIES: $c_n = \frac{f^{(n)}(0)}{n!} \Rightarrow c_0 = f(0) = 0$

$f'(x) = \frac{(1+4x) - x(4)}{(1+4x)^2} \Rightarrow f'(0) = 1 \Rightarrow c_1 = 1$

$f''(x) = \frac{d}{dx}\left[(1+4x)^{-2}\right] = (-2)(4)(1+4x)^{-3} \Rightarrow f^{(2)}(0) = -8 \Rightarrow c_2 = \frac{-8}{2} = -4$

$f^{(3)}(x) = (-2)(4)(-3)(4)(1+4x)^{-4} = 3! \cdot 4^2 (1+4x)^{-4} \Rightarrow c_3 = 4^2$

$f^{(4)}(x) = -4!\, 4^3 (1+4x)^{-5} \Rightarrow c_4 = -4^3 \ldots$

$n =$ 0, 1, 2, 3 $\longrightarrow$ n IS THE EXPONENT OF 4

THEN $M(f) = 0 + x + (-4)x^2 + 4^2 x^3 + (-4^3)x^4 + \ldots$

$= \sum_{n=0}^{\infty} (-1)^n 4^n x^{n+1} \Rightarrow$ (B)

Note that one could manipulate the geometric series, writing $f(x) = x\,\frac{1}{1-(-4x)}$

9. Find the coefficient of x^4 in the Maclaurin series for $f(x) = x\cos\left(x^3\right)$.

$f'(x) = \cos x^3 - x \cdot 3x^2 \sin x^3 = \cos x^3 - 3x^3 \sin x^3$

$f''(x) = -3x^2 \sin x^3 - 3\left(3x^2 \sin x^3 + x^3 3x^2 \cos x^3\right) = -12x^2 \sin x^3 - 9x^5 \cos x^3$

$= -3x^2\left(4 \sin x^3 + 3x^3 \cos x^3\right)$

$f^{(3)}(X) \overset{CCB}{=} 3x\left(9x^6 \sin x^3 - 27x^3 \cos x^3 - 8 \sin x^3\right)$

$f^{(4)}(x) \overset{CCB}{=} 3\left(27x^9 \cos x^3 + 144x^6 \sin x^3 - 132x^3 \cos x^3 - 8 \sin x^3\right)$

THEN $f^{(4)}(0) = 0$ AND $C_4 = \frac{1}{4!} f^{(4)}(0) = 0$.

10. Find the first four terms in the Maclaurin series for $f(x) = xe^{-x}$.

BECAUSE $f^{(0)}(0) = 0$, THE FIRST FOUR TERMS WILL BE FROM C_1 TO C_4:

$f^{(1)}(x) = e^{-x} + x(-1)e^{-x} = e^{-x} - xe^{-x} = e^{-x}(1-x) = -e^{-x}(x-1)$

$f^{(2)}(x) \overset{CCB}{=} e^{-x}(x-2)$; $f^{(3)}(x) = -e^{-x}(x-3)$; $f^{(4)}(x) = e^{-x}(x-4)$

THEN $f^{(n)} = (-1)^{n+1} n$ AND $M(f) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n-1)!} x^n$

HERE:

$x - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4$