MAT 421 – Exam 3 · Fall 2013

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Instructions. Complete 10 out of the following 18 exercises, with at least 4 of them numbered 11 or above. Each exercise is worth 10 points.

SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

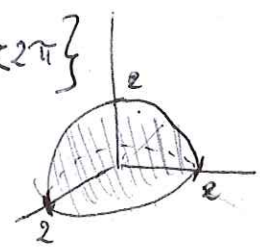
1. Evaluate the iterated integral $\int_0^{\pi}\int_0^1 x\sin y\,dx\,dy$.

$$= \int_0^1 x\,dx \int_0^{\pi} \sin y\,dy = \int_0^1 x\left[-\cos y\right]_0^{\pi} dx = \int_0^1 x(2)\,dx = \left[x^2\right]_0^1$$

$$= 1$$

2. Evaluate the triple integral $\iiint_E \sqrt{x^2+y^2+z^2}\,dV$ in spherical coordinates, where E is the solid bounded by the hemisphere $z=\sqrt{4-x^2-y^2}$ and the plane $z=0$.

$$E = \left\{(\rho,\theta,\varphi)\in\mathbb{R}^3 \mid 0\le\rho\le 2,\ 0\le\varphi\le\frac{\pi}{2},\ 0\le\theta<2\pi\right\}$$



"RADIUS OF HEMISPHERE" $=\sqrt{4}=2$

NOTE: $f(x,y,z)=\sqrt{x^2+y^2+z^2} = \|(x,y,z)\|$

$f(T(\rho,\theta,\varphi)) = \rho$ THEN THE INTEGRAL MUST BE THE VOLUME OF E.

$$T:\begin{cases} x=\rho\sin\varphi\cos\theta \\ y=\rho\sin\varphi\sin\theta \\ z=\rho\cos\varphi\end{cases} \Rightarrow \text{JACOBIAN} = \rho^2\sin\varphi$$

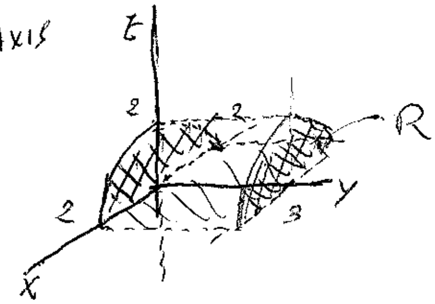
$$\iiint_E f(x,y,z)\,dV = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}}\int_0^2 \rho(\rho^2\sin\varphi)\,d\rho\,d\varphi = 2\pi\left(\frac{2^4}{4}\right)\int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi =$$

$$= 2\pi(4)(1) = 8\pi$$

3.Evaluate $\iint_R \sqrt{4-x^2}\, dA$ where $R = [-2, 2] \times [0, 3]$ by first identifying it as the volume of a solid.

VOLUME OF SEMI CIRCULAR CYLINDER WITH Y-AXIS

THEREFORE VOLUME $= \frac{1}{2}\left(\pi 2^2\right)\cdot(3) = 6\pi$



$$\iint_R \sqrt{4-x^2}\, dA = \int_0^3 dy \int_{-2}^{2} \sqrt{4-x^2}\, dx =$$

$$= 3\int_{-2}^{2} \sqrt{4-x^2}\, dx = 6\int_0^2 \sqrt{4-x^2}\, dx = 6\int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2 u}\,(2\cos u)\, du =$$

$$\boxed{x = 2\sin u \Rightarrow dx = 2\cos u\, du \quad x=0 \Rightarrow u=0\ ;\ x=2 \Rightarrow \sin u = 1 \Rightarrow u = \frac{\pi}{2}}$$

$$= 6\int_0^{\frac{\pi}{2}} 4\cos^2 u\, du =$$

$$= 6(4)\left(\frac{\pi}{4}\right) = 6\pi \quad \checkmark$$

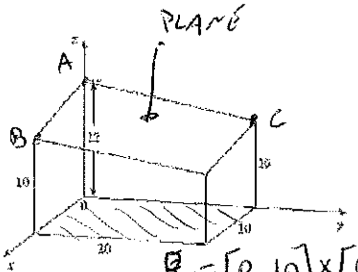
4.Calculate the iterated integral $\int_0^{\ln 3}\int_0^{\ln 7} e^{x-y}\, dx\, dy$. $= \int_0^{\ln 3} e^{-y}\, dy \cdot \int_0^{\ln 7} e^{x}\, dx =$

NOTE $f(x,y) = g(x)\, g(y) = e^{x} e^{-y}$

$$= \left[-e^{-y}\right]_0^{\ln 3} \cdot \left[e^{x}\right]_0^{\ln 7} = \left(1 - e^{-\ln 3}\right)\left(e^{\ln 7} - 1\right) = \left(1 - \tfrac{1}{3}\right)(7-1)$$

$$= \frac{2}{3}\cdot 6 = 4$$

5. A greenhouse is shown below. It is 10 ft wide and 20 ft long and has a roof that is 12 ft high at one corner and 10 ft high at each of the adjacent corners. Find the volume of the greenhouse.



$R = [0,10]\times[0,20]$

$$\text{VOLUME} = \iint_R -\frac{1}{5}x - \frac{1}{10}y + 12\, dV = \int_0^{10}\int_0^{20} -\frac{1}{5}x - \frac{1}{10}y + 12\, dy\,dx$$

$$= \int_0^{10} \left[-\frac{1}{5}xy - \frac{1}{20}y^2 + 12y\right]_0^{20} dx = \int_0^{10} -4x - 20 + 240\, dx = \left[-2x^2 + 220x\right]_0^{10} = \boxed{2000 \text{ FT}^3}$$

ANS.

PLANE THROUGH $A(0,0,12)$, $B(10,0,10)$, $C(0,20,10)$

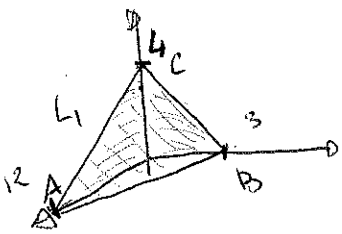
$$\text{DIRECT. VECT} = \vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 10 & 0 & -2 \\ 0 & 20 & -2 \end{vmatrix} = \langle 40, 20, 200\rangle =$$

$= 20\langle 2, 1, 10\rangle \Rightarrow \langle 2,1,10\rangle \Rightarrow$ EQ. OF PLANE: $2(x-0) + 1(y-0) + 10(z-12) = 0 \Rightarrow 2x + y + 10z - 120 = 0$

$\Rightarrow z = -\frac{1}{5}x - \frac{1}{10}y + 12$

SETUP

6. Use a double integral to find the volume of the solid bounded by the planes $x + 4y + 3z = 12$, $x = 0$, $y = 0$, and $z = 0$.



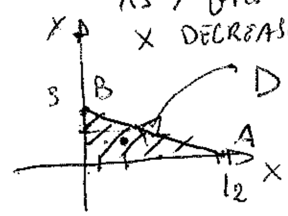
INTERSECTIONS ALONG AXIS FOR PLANE $z = 4 - \frac{1}{3}x - \frac{4}{3}y$

A: ALONG $y=0, z=0 \Rightarrow x = 12$

B: ALONG $x=0, z=0 \Rightarrow y = 3$ $\Big]\to$ LINE AB: $x = 12-4y$, $z=0$

C: ALONG $x=0, y=0 \Rightarrow z = 4$

AS Y GROWS X DECREASES



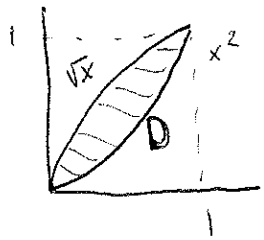
$D = \{0 \le y \le 3,\ 12-4y \le x \le 0\}$

$$\text{VOLUME} = \iint_D \text{"PLANE"}\, dV = \int_0^3 dy \int_{\frac{12-4x}{3}}^{0} 4 - \frac{1}{3}x - \frac{4}{3}y\, dx$$

$$= \int_0^3 -\left[4x - \frac{1}{6}x^2 - \frac{4}{3}xy\right]_0^{12-4y} dx = -\int_0^3 \underbrace{4(12-4y) - \frac{1}{6}(12-4y)^2 - \frac{4}{3}y(12-4y)}_{48 - 16y - \frac{8}{3}y^2 - 24 + 16y - 16y + \frac{16}{3}y^2}\, dy$$

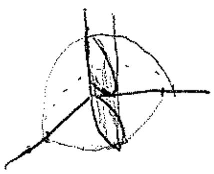
$$= -\int_0^3 24 - 16y - \frac{8}{3}y^2\, dy = -8\left[3y - y^2 - \frac{1}{9}y^3\right]_0^3 = -8(9 - 9 - 3) = 24$$

7. Find the volume of the solid under the surface $z = x^2 + y^2$ and lying above the region $\{(x,y) \mid 0 \le x \le 1, x^2 \le y \le \sqrt{x}\}$.



$$V = \iint_D z\,dA = \int_0^1 \int_{x^2}^{\sqrt{x}} x^2 + y^2 \,dy\,dx =$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_{x^2}^{\sqrt{x}} dx = \int_0^1 x^{\frac{5}{2}} + \frac{x^{\frac{3}{2}}}{3} - x^4 - \frac{x^6}{3}\,dx =$$



$$= \left[\frac{2}{7} x^{\frac{7}{2}} + \frac{1}{3}\left(\frac{2}{5}\right) x^{\frac{5}{2}} - \frac{1}{5} x^5 - \frac{1}{21} x^7\right]_0^1 = \frac{2}{7} + \frac{2}{15} - \frac{1}{5} - \frac{1}{21} =$$

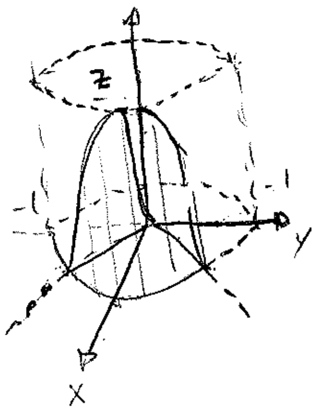
$$= \frac{6}{35}$$

8. Find the volume bounded above by the surface $z = x^2 - y^2$, $x \ge 0$, below by the xy-plane, and laterally by the cylinder $x^2 + y^2 = 1$.

IT IS EASIER TO EXPRESS THE DOMAIN D IN POLAR COORD.

$$D = \left\{ 0 \le r \le 1, -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \right\}.$$

THE SOLID IS SYMMETRIC WITH RESPECT TO THE XZ-PLANE, THEN:

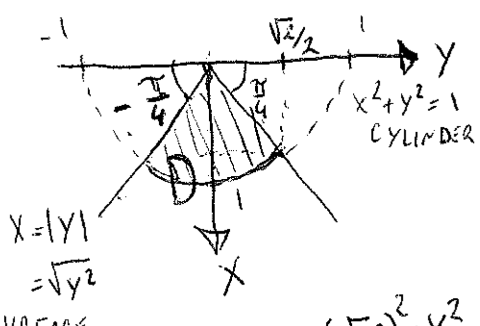


$$V = \iint_D z\,dA = \iint_D \left((r\cos\theta)^2 - (r\sin\theta)^2\right) r\,dr\,d\theta$$

$$= 2\int_0^1 \int_0^{\frac{\pi}{4}} r^3 (\cos^2\theta - \sin^2\theta)\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\cos 2\theta\,d\theta \cdot \int_0^1 r^3\,dr$$

XY - TRACES



$$= \left[\sin 2\theta\right]_0^{\frac{\pi}{4}} \cdot \left[\frac{r^4}{4}\right]_0^1$$

$$= 1 \cdot \frac{1}{4} = \frac{1}{4}$$

INTERSECTIONS: $(\sqrt{y^2})^2 + y^2 = 1$

$\Rightarrow 2y^2 = 1 \Rightarrow y = \pm\frac{\sqrt{2}}{2}$

9. Find the Jacobian of the transformation $x = 2u,\ y = 3v^2,\ z = 4w^3$.

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 6v & 0 \\ 0 & 0 & 12w^2 \end{vmatrix} = 2(6v)(12w^2)$$

$$= 144\, v w^2$$

10. Use the change of variables $x = \sqrt{2}u - \sqrt{\frac{2}{3}}v,\ y = \sqrt{2}u + \sqrt{\frac{2}{3}}v$ to evaluate $\iint_R (x^2 - xy + y^2)\, dA$, where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$.

$$T(u,v) = \left(\sqrt{2}u - \sqrt{\tfrac{2}{3}}v,\ \sqrt{2}u + \sqrt{\tfrac{2}{3}}v\right)$$

NOTE: $R:\ (x-y)^2 + xy = 2$

$$\Rightarrow T^{-1}(R):\ \left(-2\sqrt{\tfrac{2}{3}}v\right)^2 + \left(\sqrt{2}u\right)^2 + - \left(\sqrt{\tfrac{2}{3}}v\right)^2 = 2 \iff \frac{8}{3}v^2 + 2u^2 - \frac{2}{3}v^2 = 2 \iff u^2 + v^2 = 1$$

THEN:

MOREOVER: ON $T^{-1}(R)$ WE HAVE $f(T(u,v)) = u^2 + v^2 = 1$

$$\iint_R x^2 - xy + y^2\, dA = \iint_{T^{-1}(R)} 1\, d(T^{-1}A) = \iint_{\substack{\text{CIRCLE} \\ u^2+v^2=1}} 1 \frac{4}{\sqrt{3}}\, du\,dv = \text{"AREA OF CIRCULAR CYLINDER"} = \pi(1)^2 \frac{4}{\sqrt{3}}$$

$$= \frac{4\pi}{\sqrt{3}} \approx 7.26$$

JACOBIAN: $\dfrac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u = \sqrt{2}\left(\sqrt{\tfrac{2}{3}}\right) - \left(-\sqrt{\tfrac{2}{3}}\right)\sqrt{2} = \dfrac{4}{\sqrt{3}}$

11. Find a function of $f(x,y)$ such that $\nabla f = \mathbf{F}(x,y) = \langle xy^2+2, x^2y+2 \rangle$.

CHECK IT IS CONSERVATIVE: $F = \langle P, Q \rangle$ AND $P_y = Q_x$: $2XY = 2XY$ ✓

$\nabla f = \langle f_x, f_y \rangle = \langle P, Q \rangle \Rightarrow f = \int P\,dx = \int xy^2+2\,dx = \frac{x^2y^2}{2} + 2x + g(y)$

$Q = f_y \Rightarrow \cancel{x^2y} + 2 = \frac{\partial}{\partial y}\left[\frac{x^2y^2}{2} + 2x + g(y)\right] = \cancel{x^2y} + 0 + g'(y) \Rightarrow$

$\Rightarrow g = \int 2\,dy = 2y + C \Rightarrow f = \frac{x^2y^2}{2} + 2x + 2y + C$

12. Find the work done by the force $\mathbf{F} = (2x+y)\mathbf{i} + (xy)\mathbf{j}$ in moving an object from $(1,0)$ to $(2,3)$ along the path C given by $x = t+1$, $y = 3t$.

DISPLACEMENT: $\vec{r}(t) = \langle t+1, 3t \rangle$, $0 \le t \le 1$

$W = \int_0^1 \vec{F} \cdot \vec{r}'\,dt = \int_0^1 \langle 2(t+1)+3t, (t+1)(3t) \rangle \cdot \langle 1, 3 \rangle\,dt$

$= \int_0^1 2(t+1) + 3t + 9t^2 + 9t\,dt$

$\int_0^1 14t + 2 + 9t^2\,dt \; \cancel{9t^2)}dt = \left[2t + 7t^2 + 3t^3\right]_0^1 = 2 + 7 + 3 = 12$

$= \int_0^1 \cancel{2t + 5t^2 + 9t^3 + 2 + 5t + 9t^2\,dt} = \int_0^1 \cancel{2 + 7t + 14t^2 + 9t^3}\,dt$

$= \cancel{\left[2t + \frac{7}{2}t^2 + \frac{14}{3}t^3 + \frac{9}{4}t^4\right]_0^1 = 2 + \frac{7}{2} + \frac{14}{3} + \frac{9}{4} = \frac{149}{12}}$ JOULES

13. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = (y+z)\,\mathbf{i} - x^2\mathbf{j} - 4y^2\mathbf{k}$, and the curve C is given by the vector function $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k},\ 0 \le t \le 1$.

$$F|_C = \langle t^2+t^4, -t^2, -4t^4 \rangle \;;\; \vec{r}\,' = \langle 1, 2t, 4t^3 \rangle$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}|_C \cdot \vec{r}\,' dt = \int_0^1 t^2+t^4-2t^3-16t^7\,dt$$

$$= \left[\frac{t^3}{3} - \frac{t^4}{2} + \frac{t^5}{5} - 2t^8 \right]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - 2 = -\frac{59}{30} = -1.9\overline{6}$$

14. Determine whether or not $\mathbf{F}(x,y) = \underbrace{\left(ye^{xy} + 4x^3y\right)}_{P}\mathbf{i} + \underbrace{\left(xe^{xy} + x^4\right)}_{Q}\mathbf{j}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

$$P_y = e^{xy}(1+xy) + 4x^3 \overset{\checkmark}{=} Q_x = e^{xy}(1+xy) + 4x^3$$

$$f = \int P\,dx = \int ye^{xy} + 4x^3y\,dx = e^{xy} + x^4y + g(y) \quad \text{AND } f_y = Q \Rightarrow$$

$$\Rightarrow xe^{xy} + x^4 + g'(y) = xe^{xy} + x^4 \Rightarrow g'(y) = 0 \Rightarrow g(y) = c$$

THEN $f = e^{xy} + x^4y + c$

15. Find a function f such that $\mathbf{F} = \nabla f$ and use it to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C. $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$; C is the arc of the curve $y = x^4 - x^3$ from $(1,0)$ to $(2,8)$.

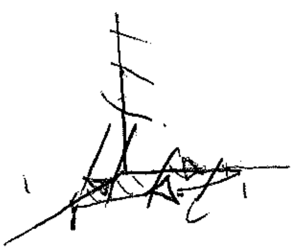
$$P_y = Q_x = 1 \Rightarrow f = \int P\,dx = \int y\,dx = xy + g(y) \Rightarrow f_y = Q:$$

$$x + g'(y) = x \Rightarrow g'(y) = 0 \Rightarrow g(y) = c \Rightarrow f = xy + c \; : \text{CHOOSE } c = 0$$

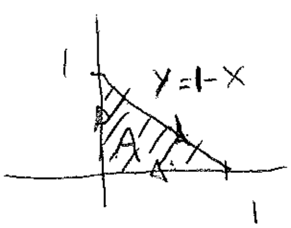
$$\vec{F} = \nabla f \Rightarrow \int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A) = 2 \cdot 8 - 1 \cdot 0 = 16$$

16. Evaluate the line integral $\oint_C xy^2\,dx - yx^2\,dy$ around the triangle with vertices $(1,0)$, $(0,1)$, and $(0,0)$ with clockwise orientation.

NOTE: $\vec{F} = \langle xy^2, -yx^2 \rangle$ IS NOT CONSERVATIVE.



$C = -\partial A$ BOUNDARY LINE, WITH REVERSE ORIENTATION, OF THE TRIANGULAR REGION $A = \{ 0 \le x \le 1,\ 0 \le y \le 1-x \}$



$$\oint_C P\,dx + Q\,dy = \iint_{\partial A \to -\partial A} P_y - Q_x \,dx\,dy$$

(NOTE: $P_y - Q_x$ INSTEAD OF $Q_x - P_y$)

$$= \int_0^1 dx \int_0^{1-x} 2xy - (-2yx)\,dy = 2\int_0^1 \left[\frac{xy^2}{2}\right]_0^{1-x} dx$$

$$\cancel{= 2\int_0^1 x^3\,dx = \frac{1}{2}\left[x^4\right]_0^1 = \frac{1}{2}}$$

$$(1-x)^2 x\,dx = 2\int_0^1 x^3 - 2x^2 + x\,dx = 2\left[\frac{x^4}{4} - \frac{2}{3}x^3 + \frac{x^2}{2}\right]_0^1$$

$$= 2\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) = \frac{1}{6}$$

17. Find (a) the curl and (b) the divergence of the vector field $\mathbf{F}(x, y, z) = x^2y\mathbf{i} + yz^2\mathbf{j} + zx^2\mathbf{k}$.

$$\text{div}(\vec{F}) = \vec{\nabla} \cdot \vec{F} = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle x^2y, yz^2, zx^2 \rangle = 2xy + z^2 + x^2$$

$$\text{CURL}(\vec{F}) = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \partial x & \partial y & \partial z \\ x^2y & yz^2 & zx^2 \end{vmatrix} = \langle (0 - 2zy), \cancel{+}(2xz - 0), (0 - x^2) \rangle$$

$$= \langle -2yz, +2xz, -x^2 \rangle$$

$$= -\langle 2yz, 2xz, x^2 \rangle$$

18. According to Green's Theorem, the line integral $\int_C y^2dx + x^2dy$ over a positively oriented, piecewise-smooth, simple closed curve C is equal to the double integral $\iint_D f(x, y)\, dA$ over the region D bounded by C. Find the function $f(x, y)$.

$$\int_C y^2 dx + x^2 dy = \int_C \vec{F} \cdot d\vec{r} \quad \text{so} \quad \vec{F} = \langle y^2, x^2 \rangle = \langle P, Q \rangle$$

THEN $f(x, y) = Q_x - P_y$ BY GREEN'S THEOREM.

THEREFORE $f(x, y) = 2x - 2y$