Math 221- Spring 2010 - Test 4

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Instructions. You **can not** use a graph to justify your answer.

SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. (*10 pts*) (#20, page 209) Find any critical numbers (if any) of the function $g(x) = 4x^2(3^x)$.

$$g'(x) = 4\cdot\left(2x\cdot 3^x + x^2\cdot \ln 3\cdot 3^x\right) = 4x\,3^x\left(2 + x\ln 3\right)$$

$g'(x)$ IS NEVER UNDEFINED

$$g'(x) = 0 \iff 4x\,3^x(2+\ln 3\cdot x) = 0 \Rightarrow \begin{cases} 3^x = 0, \text{ NEVER} \\ x = 0 \\ 2 + \ln 3\cdot x = 0 \rightarrow x = -\frac{2}{\ln 3}\end{cases}$$

THERE ARE ONLY TWO CRIT. #'S: $x = 0$ AND $x = -\frac{2}{\ln 3} \approx -1.82$

2. (*10 pts*) (#34, page 209) Find the relative extrema (if any) and the absolute extrema of the function $y = x\ln(x+3)$ on the closed interval $[0, 3]$.

DOMAIN OF y: $x + 3 > 0 \rightarrow x > -3$. THIS FUNCTION IS CONTINUOUS ON $[0, 3]$ THEREFORE y HAS ABSOLUTE EXTREMA.
A CRITICAL NUMBER WILL BE AN ABS. EXTREMA ONLY IF IT IS A REL.

$$y' = 1\cdot \ln(x+3) + x\cdot\frac{1}{x+3} = \frac{(x+3)\ln(x+3) + x}{x+3}$$

y' IS NEVER UNDEFINED ON $[0, 3]$.

$y' = 0$ DOES NOT ADMIT SOLUTIONS ON $[0, 3]$ (EITHER BY GRAPH OR) BECAUSE $(x+3)\ln(x+3) + x > 0$, INDEED, ON $[0, 3]$ THE TERM $(x+3)\ln(x+3)$ IS POSITIVE.
THEREFORE THERE IS NOT ANY CRIT. # AND THUS NONE REL. EXTREMA.

AS SAID, $y' > 0$ AND y IS INCREASING THUS WE HAVE $\boxed{\text{THE ABS. MIN AT } (0, f(0) = 0)}$ AND THE $\boxed{\text{ABS. MAX AT } (3, f(3) = 3\ln 6 \approx 5.37)}$.

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3. (*15 pts*) (#50, page 226) Find the domain of the function $f(x) = x + 2\sin x$, then use the first derivative test in order to find (if any) the relative extrema of $f(x)$.

$f(x)$ IS DEFINED FOR ALL REAL NUMBERS: $(-\infty, \infty)$

$f'(x) = 1 + 2\cos x$ (NEVER UNDEFINED)

$$f'(x) = 0 \to 1 + 2\cos x = 0 \to \cos x = -\frac{1}{2} \to \begin{cases} x = \frac{2}{3}\pi + 2n\pi \\ x = \frac{4}{3}\pi + 2n\pi \end{cases}$$

BECAUSE $f'(x)$ IS PERIODIC, WE CAN APPLY THE TEST ON $[0, 2\pi]$

$2\pi/3$ $\quad$ $4\pi/3$

x	0	$\pi/2$	2π
$f'(x)$ SIGN	+	−	+
$f(x)$ INCR			

MAX $\quad$ MIN

REL MAX AT $\left(\frac{2}{3}\pi + 2n\pi, \frac{2}{3}\pi + 2n\pi + \sqrt{3}\right)$

REL MIN AT $\left(\frac{4}{3}\pi + 2n\pi, \frac{4}{3}\pi + 2n\pi - \sqrt{3}\right)$

WHERE n IS ANY INTEGER

4. (*10 pts*) (#20, page 235) Find (if any) the inflection points of the function $f(x) = \dfrac{x+1}{\sqrt{x}}$.

WE NEED TO CHECK CONCAVITY CHANGE, THAT IS SIGN CHANGE OF $f''(x)$.

REWRITE: $f(x) = \dfrac{x}{x^{1/2}} + \dfrac{1}{x^{1/2}} = x^{1/2} + x^{-1/2}$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-3/2} = \frac{1}{2}\left(x^{-1/2} - x^{-3/2}\right)$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}x^{-3/2} + \frac{3}{2}x^{-5/2}\right) = \frac{1}{4}x^{-5/2}(-x+3)$$ THEN

$$f''(x) = \frac{1}{4}\,\frac{3-x}{x^{5/2}}$$

UNDEFINED: $x = 0$ OUTSIDE DOMAIN!!!

ZERO: $3 - x = 0 \to x = 3$

NOTICE THAT DOMAIN OF $f(x)$ IS $x > 0$ (BECAUSE $x \geq 0$, $x \neq 0$).

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x	1	4
$f''(x)$ SIGN	+	−
$f(x)$ CONC.	∪	∩

CHANGE OF SIGN FOR $f''(x)$, IS CHANGE OF CONCAVITY FOR $f(x)$

$f(x)$ HAS ONE INFLECTION POINT

A $\left(3, f(3) = \dfrac{4}{\sqrt{3}}\right)$

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5. (*10 pts*) (#50, page 235) Find the domain of the function $f(x) = xe^{-x}$, then use the second derivative test in order to find (if any) the relative extrema of $f(x)$.

DOMAIN: $(-\infty, \infty)$

$f'(x) = e^{-x} + xe^{-x}\cdot(-1) = e^{-x}(1-x)$

$f''(x) = e^{-x}\cdot(-1) - \frac{d}{dx}[xe^{-x}] = -e^{-x} - e^{-x}(1-x) = -e^{-x}(2-x)$

CRIT. #'S: $f'(x)$ IS NEVER UNDEFINED. $f'(x) = 0 \Rightarrow \begin{cases} e^{-x} = 0, \text{ NEVER} \\ 1-x = 0 \Rightarrow x = 1 \end{cases}$

$f''(1) = -1^{-1}(2-1) < 0$ $\cap$

REL. MAX AT $(1, f(1) = e^{-1} \approx .368)$

6. Find the following limits, if possible. Write the known limit or the rule for horizontal asymptote that you use.

(a) (*10 pts*) (#18b, page 245) $\lim\limits_{x\to\infty} \frac{3-2x}{3x-1} = \frac{-2}{3} = -\frac{2}{3}$

SAME DEGREE $\to$ QUOTIENT OF LEADING TERMS

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(b) (*10 pts*) (#20a, page 245) $\lim_{x\to\infty} \frac{5x^{3/2}}{4x^2+1} =$ 0

DEGREE NUM < DEGREE DEN.

(c) (*10 pts*) (#28, page 245) $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} =$ 1

$$\frac{x}{\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}} = \frac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}} = \frac{1}{\sqrt{1+\underbrace{\frac{1}{x^2}}_{\to 0}}} \to \frac{1}{1}$$

(d) (*10 pts*) (#34, page 245) $\lim_{x\to\infty} \cos\frac{1}{x} =$ $\cos\left(\lim_{x\to\infty}\frac{1}{x}\right) = \cos 0 = 1$

CONTINUITY

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7. (*20 pts*) (#60, page 269) Fifty elk are introduced into a game preserve. It is estimated that their population will increase according to the model

$$p(t) = \frac{250}{1 + 4e^{-t/3}},$$

where t is measured in years. a) ✓ At what rate is the population increasing when $t = 2$? b) ✓ After how many years is the population increasing *most rapidly*? c) What is the limiting size of this population of elks?

a) ASKED FOR $p'(2)$: $p'(t) = 250 \cdot \frac{d}{dt}\left[\left(1+4e^{-t/3}\right)^{-1}\right] =$

$$= 250 \cdot (-1)\left(1+4e^{-t/3}\right)^{-2} \cdot \left(4e^{-t/3}\left(-\tfrac{1}{3}\right)\right)$$

$$p'(t) = \frac{1000}{3} \frac{e^{-t/3}}{\left(1+4e^{-t/3}\right)^2}$$

$$\boxed{\begin{aligned} &c)\ L.S. = \lim_{t\to\infty} p(t) = \frac{250}{1+4\cdot 0} = 250 \text{ ELKS} \\ &\lim_{t\to\infty} e^{-t/3} = \sqrt[3]{\lim_{t\to\infty} e^{-t}} = \sqrt[3]{0} = 0 \end{aligned}}$$

$$p'(2) \approx 18.35$$

AFTER 2 YEARS THE POPULATION IS GROWING AT (ABOUT) A RATE OF 18 ELK PER YEAR.

b) "MOST RAPIDLY" MEANS AT MAXIMUM RATE. WE NEED REL MAX FOR $j(t) = p'(t)$.

$$j'(t) = p''(t) = \frac{1000}{3} \frac{e^{-t/3}\cdot\left(-\frac{1}{3}\right)\cdot\left(1+4e^{-t/3}\right)^2 - e^{-t/3}\cdot 2\left(1+4e^{-t/3}\right)\left(-\frac{4}{3}e^{-t/3}\right)}{\left(1+4e^{-t/3}\right)^4}$$

$$= \frac{1000}{3} \frac{-\frac{e^{-t/3}}{3}\left(1+4e^{-t/3}\right)\left(1+4e^{-t/3} - 8e^{-t/3}\right)}{\left(1+4e^{-t/3}\right)^4}$$

$$= -\frac{1000}{9} \frac{e^{-t/3}\left(1-4e^{-t/3}\right)}{\left(1+4e^{-t/3}\right)^3} = 0 \to 1 - 4e^{-t/3} = 0 \to e^{-t/3} = \frac{1}{4} \to$$

NEVER UNDEFINED

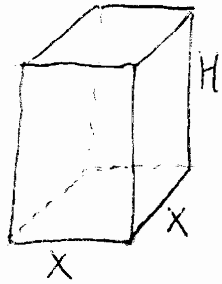
$$\to -t/3 = \ln\left(\tfrac{1}{4}\right) \to t = -3\ln\left(\tfrac{1}{4}\right) \approx 4.16$$

t	4	4.16	5
$j'(t)$	+		−
$j(t)$	$\nearrow$	MAX	$\searrow$

✓

AT THE BEGINNING OF THE FOURTH YEAR THE POPULATION IS INCREASING MOST RAPIDLY

8. (*15 pts*) (#22, page 266) Determine the dimensions of a rectangular solid (with a square base) with maximum volume if its surface area is 337.5 square centimeters.



PRIMARY EQUATION: $V = X^2 H$

SECONDARY EQUATION: $S = 337.5 \rightarrow 2X^2 + 4XH = 337.5$

$\rightarrow H = \dfrac{337.5 - 2X^2}{4X}$ PLUG IN PRIM.

PRIMARY EQ: $V = \dfrac{1}{4}\left(337.5X - 2X^3\right) \rightarrow V' = \dfrac{1}{4}\left(337.5 - 6X^2\right)$

$V' = 0 \rightarrow 337.5 - 6X^2 = 0 \rightarrow X = \pm\sqrt{337.5/6} = \pm 7.5$ cm REJECTED

2ND DERIVATIVE TEST: $V'' = \dfrac{1}{4}\cdot(-6)\cdot 2X \rightarrow V''(7.5) < 0$ MAX

$H(7.5) = \dfrac{337.5 - 2(7.5)^2}{4\cdot 7.5} = 7.5$. A CUBE OF SIDE 7.5 cm

9. (*20 pts*) (#18, page 276) Determine the differentials $d[uv]$, $d[e^u]$, and $d[\cos u]$. Then compute the differential of the function $y = e^{-0.5}\cos(4x)$.

$d[uv] = \dfrac{d}{dx}[uv]\cdot dx = (uv' + vu')dx = u(v'dx) + v(u'dx)$

$= u\,dv + v\,du$

$d[e^u] = \dfrac{d}{dx}[e^u]\cdot dx = e^u u' dx = e^u du$

$d[\cos u] = \dfrac{d}{dx}[\cos u]\cdot dx = (-\sin u)\, u' dx = -\sin u\, du$

THERE WAS A TYPO, WHICH MAKES THIS SIMPLER *.

$dy = \dfrac{d}{dx}\left[e^{-0.5}\cos(4x)\right]dx = e^{-0.5}\dfrac{d}{dx}[\cos(4x)]\,dx$

$= e^{-0.5}(-4)\sin(4x)\,dx$

$\left(\text{OR } -e^{-0.5}4\sin(4x)\,dx\right)$

* NOTE: IF AS IN THE BOOK

$\dfrac{d}{dx}\left[e^{-0.5x}\cos(4x)\right] = \left(-\dfrac{1}{2}e^{-0.5x}\cos(4x) - 4e^{-0.5}\sin(4x)\right)dx$

$= -\dfrac{1}{2}e^{-0.5x}\left(\cos(4x) + 8\sin(4x)\right)dx$