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Math 321- Spring 2013 - Exam2

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Instructions. Technology and instructor's notes (including the formula sheets from our book) are allowed on this exam. Each problem is worth 10 points. **If you use notes or formula sheets, make a reference. When using technology describe which commands (or keys typed) you used or print out your worksheet.**

1. The radioactive isotope Tshirt-101 has a half-life of 7 days. Suppose we have an initial amount of 500 mg.
   (a) Use a natural exponential decay ODE to model this isotope amount after t days.
   (b) What is the amount of Tshirt-101 remaining after 10 days?

(a) ODE: $P' = KP \longrightarrow$ SOLUTION: $P = P_0 e^{Kt}$.

PLUG INITIAL CONDITION: $P_0 = P(0) = 500 \longrightarrow P = 500 e^{Kt}$

PLUG HALF-LIFE: $250 = P(7) \longrightarrow \frac{250}{250} = \frac{500}{250} e^{7K} \longrightarrow$

$\longrightarrow \ln(e^{7K}) = \ln(\frac{1}{2}) \longrightarrow 7k = \ln(\frac{1}{2}) \longrightarrow K = \frac{1}{7}\ln(2^{-1}) = -\frac{\ln 2}{7} \approx -.099$

$P = 500 e^{(-\ln 2)\frac{1}{7}t} \longrightarrow P = 500(2^{-\frac{t}{7}})$ (OR $P = 500 e^{-.099t}$)

(b) $P(10) = 500(2^{-10/7}) \approx 500 e^{-.099(10)} \approx 185.79$ OR 186 mg

2. Consider the ODE

$$\frac{dy}{dx} = y^2 - 2xy. \qquad (1)$$

   (a) Find the general solution of (1).
   (b) Solve the IVP given by (1) and the condition $y(1) = 3$.

IMPLICIT

(a) SOLVEODE$[y^2 - 2xy] \longrightarrow \frac{1}{Y} = e^{x^2}\left(C - \frac{\sqrt{\pi}}{2}\mathrm{erf}(x)\right) \longrightarrow$

EXPLICIT

$\longrightarrow Y = e^{-x^2}\left(C - \frac{\sqrt{\pi}}{2}\mathrm{erf}(x)\right)^{-1}$

SINGULAR SOLUTION: $Y' = Y(Y - 2X) \Rightarrow Y = 0$ SING. SOL. $\longrightarrow$ GEN. SOL.

(b) PLUG $Y=3$ AND $X=1$; $3 = e^{-1}\left(C - \frac{\sqrt{\pi}}{2}\mathrm{erf}(1)\right) \longrightarrow C \approx 0.869$

[NOTE THAT $\mathrm{erf}(1) = \frac{2}{\sqrt{\pi}}\int_0^1 e^{-t^2}dt \approx .842701$]

$Y = e^{-x^2}(.869 - .886\,\mathrm{erf}(x))^{-1}$

3. Find the orthogonal trajectories of the family of curves $y = kx^3$. Then draw several members of each family on the same coordinate plane. (You can attach a printout or upload it in EagleWeb.)

I) SLOPE OF GIVEN FAMILY: $y' = k\,3x^2 \Rightarrow y' = 3kx^2$

$y = kx^3 \Rightarrow k = \dfrac{y}{x^3}$

$\Rightarrow y' = 3\left(\dfrac{y}{x^3}\right)x^2 \Rightarrow y' = 3\dfrac{y}{x}$ (OUR $y' = F(x,y)$)

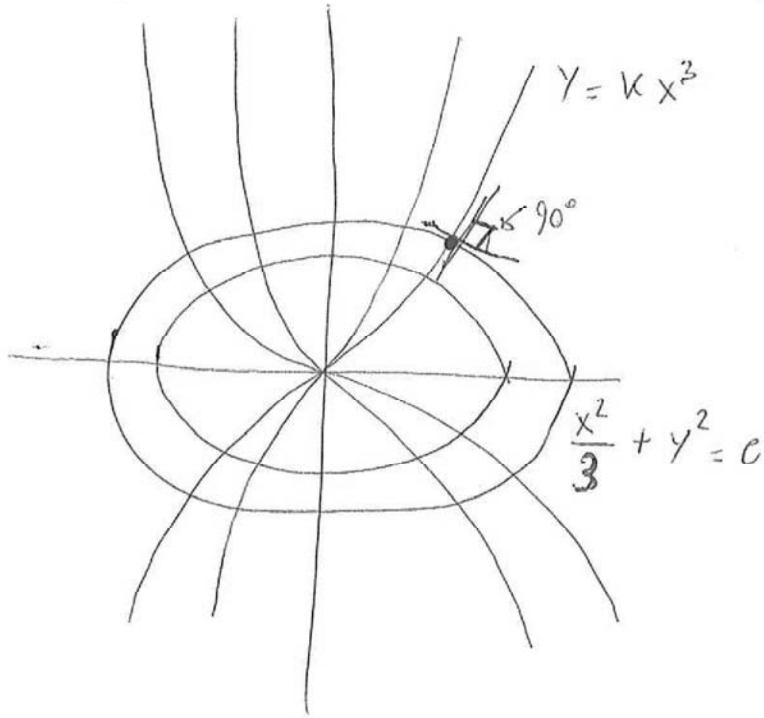
II) SLOPE OF ORTHOGONAL TRAJECTORIES $= -\dfrac{1}{F}$: $y' = -\dfrac{x}{3y}$

III) SOLVE ODE: $\dfrac{dy}{dx} = -\dfrac{1}{3}\dfrac{x}{y} \Rightarrow \int y\,dy = \int -\dfrac{1}{3}x\,dx \Rightarrow$

$\Rightarrow \dfrac{y^2}{2} = -\dfrac{1}{3}\dfrac{x^2}{2} + C_0 \Rightarrow \dfrac{x^2}{6} + \dfrac{y^2}{2} = C_0$ $(C_0 > 0)$

REPLACE $2C_0$ WITH C

$$\boxed{\frac{x^2}{3} + y^2 = C}$$



4. Suppose a population growth is modeled by the logistic differential equation with the carrying capacity 3500 and the relative growth rate $k = 0.07$ per year.

   (a) Express the logistic equation.

   (b) Express the general solution.

   (c) Express the particular solution for which $P(0) = 700$.

(a) LOGISTIC ODE: $P' = kP\left(1 - \frac{P}{M}\right)$

HERE $K = .07$ AND $M = 3500$

$\rightarrow P' = .07P\left(1 - \frac{P}{3500}\right)$

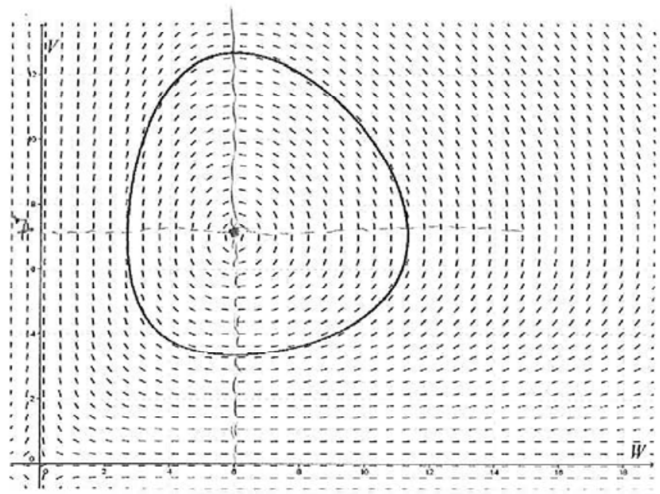
(b) GEN. SOL.: $P = \frac{M}{1 + Ae^{-Kt}} \rightarrow$

$\rightarrow P = \frac{3500}{1 + Ae^{-.07t}}$

(c) PLUG $P = 700,\ t = 0$: $700 = \frac{3500}{1 + A} \rightarrow$

$\rightarrow 1 + A = \frac{3500}{700} \Rightarrow A = 4 \Rightarrow P = \frac{3500}{1 + 4e^{-.07t}}$

5. A phase portrait of a predator-prey system is given below in which V represents the population of vampires (in hundreds) and W the population of werewolves (in hundreds).



(a) Referring to the graph, what is a reasonable non-zero equilibrium solution for the system?

(b) Write down a possible system of differential equations which could have been used to produce the given graph.

(a) NON-ZERO EQUILIBRIUM $\equiv$ "CENTER OF I-QUADRANT SLOPE FIELD"

$\equiv (6, 7)$

(b)
$$\begin{cases} \dfrac{dW}{dt} = W(7 - V) \\ \dfrac{dV}{dt} = -V(6 - W) \end{cases}$$

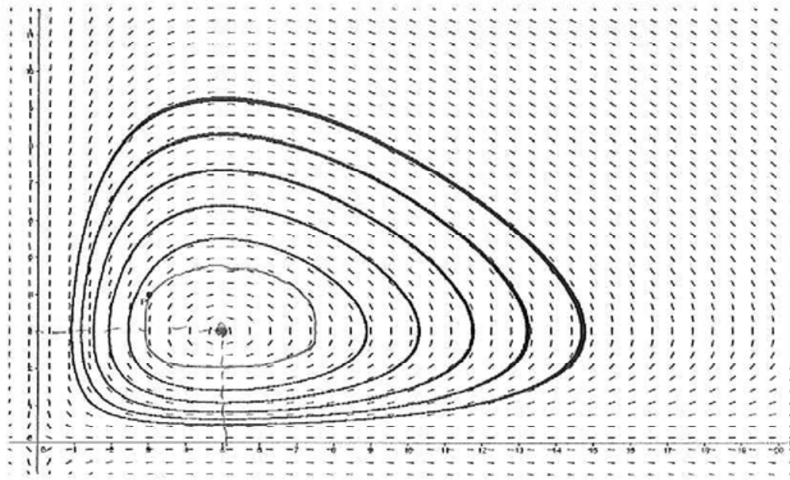
6. Consider the following predator-prey system where x and y are in millions of creatures and t represents time in years:

$$\begin{cases} \dfrac{dx}{dt} = 9x - 3xy \\ \dfrac{dy}{dt} = -10y + 2xy \end{cases}$$

(a) Show that $(5, 3)$ is the nonzero equilibrium solution.

(b) Find an expression for $\dfrac{dy}{dx}$.

(c) The direction field for the differential equation at point (6b) is given below. Locate $(5, 3)$ on the graph and sketch a rough phase trajectory through $P = (3, 4)$ indicated in the graph.



(a) $\begin{cases} \dfrac{dx}{dt} = x(9-3y) \\ \dfrac{dy}{dt} = -y(10-2x) \end{cases} \xrightarrow{\text{EQUILIBRIUM}} \begin{cases} x(9-3y) = 0 \\ -y(10-2x) = 0 \end{cases} \longrightarrow$

$\xrightarrow{\text{NON-ZERO}} \begin{cases} 9-3y=0 \rightarrow y = 3 \\ 10-2x = 0 \rightarrow x = 5 \end{cases}$

(b) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-10y+2xy}{9x-3xy}$

(c) PICTURE.