Math 221- Spring 2010 - Test 3

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Instructions. A graphing calculator, a 3″ × 5″ index card and the "unit-circle", are allowed on this examination. Point values of each problem are indicated. **Always use the appropriate wording and units of measure in your answers (when applicable).**
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Compute the first derivatives of the following functions. Show your work, applying the correct derivation rule, and give a simplified expression.

   (a) (*15 pts*) $y = \frac{e^x}{x+4}$

   QUOTIENT RULE: $y' = \dfrac{\frac{d}{dx}[e^x]\cdot(x+4) - e^x \frac{d}{dx}[x+4]}{(x+4)^2} =$

   $= \dfrac{e^x(x+4) - e^x \cdot 1}{(x+4)^2} = \dfrac{e^x(x+4-1)}{(x+4)^2}$

   $= \dfrac{e^x(x+3)}{(x+4)^2}$

   (b) (*15 pts*) $y = \ln \frac{(x-1)^3}{(2x+3)}$

   $y' = \dfrac{d}{dx}\left[\ln(x-1)^3 - \ln(2x+3)\right] = \dfrac{d}{dx}\left[3\ln(x-1)\right] - \dfrac{d}{dx}\left[\ln(2x+3)\right] =$

   $= 3\dfrac{d}{dx}\left[\ln(x-1)\right] - \dfrac{\frac{d}{dx}[2x+3]}{2x+3} = 3\dfrac{1}{x-1} - \dfrac{2}{2x+3} =$

   $= \dfrac{3(2x+3) - 2(x-1)}{(x-1)(2x+3)} = \dfrac{6x+9-2x+2}{(x-1)(2x+3)} = \dfrac{4x+11}{(x-1)(2x+3)}$

   (c) (*15 pts*) $y = 1 + \arctan(4-5x)$

   $y' = 0 + \dfrac{\frac{d}{dx}[4-5x]}{1+(4-5x)^2} = \dfrac{-5}{1+(4-5x)^2} =$

   NOT NEEDED

   $= \dfrac{-5}{1+16-40x+25x^2} = \dfrac{-5}{25x^2-40x+17}$

2. (*15 pts*) Compute the **second order derivative** of $y = \frac{3}{4}e^x + 2\cos x$.

SECOND ORDER DERIVATIVE: $y'' = \frac{d}{dx}[y']$

$y' = \frac{3}{4}\frac{d}{dx}[e^x] + 2\frac{d}{dx}[\cos x] = \frac{3}{4}e^x - 2\sin x$ →

→ $y'' = \frac{3}{4}e^x - 2\cos x$

3. (*15 pts*) Determine the point(s) (if any) at which the graph of the function $y = x + \sin x$ as a horizontal tangent line for $0 \leq x < 2\pi$.

HORIZ. TANG. LINE: $y' = 0$

$y' = 1 + \cos x$

→ $1 + \cos x = 0$ → $\cos x = -1$ →

→ $x = \pi$.

POINT: (x, y) → $y(\pi) = \pi + \sin\pi = \pi$ →

→ HORIZ. TANG. LINE AT (π, π)

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4. (*15 pts*) Write an equation of the tangent line to graph of $y^4 = y^2 + 3x^2$ at the point $(2, 2)$.

TANG. LINE AT (x_0, y_0) : $y - y_0 = y_0' \cdot (x - x_0)$

$(x_0, y_0) = (2, 2)$

TO FIND y', IMPLICIT DIFFERENTIATION: $\frac{d}{dx}[y^4] = \frac{d}{dx}[y^2 + 3x^2] \to$

$\to 4y^3 y' = 2y \cdot y' + 6x \to 4y^3 y' - 2y y' = 6x \to$

$\to 2(2y^3 - y) y' = 6x \to y' = \frac{3x}{2y^3 - y} \to y_0' = \frac{3 \cdot 2}{2 2^3 - 2} = \frac{3}{7}$

$\to y - 2 = \frac{3}{7}(x - 2) \xrightarrow[\text{NEEDED}]{\text{NOT}} y = \frac{3}{7}x + \frac{8}{7}$

5. (*15 pts*) After t years, the value of a car purchased for \$20,000 is

$$V(t) = 20,000 \left(\frac{3}{4}\right)^t.$$

At what rate is the value of the car changing after 3 years?

$V'(t) = \text{RATE} = \frac{d}{dt}[V] = 20000 \cdot \ln\frac{3}{4} \cdot \left(\frac{3}{4}\right)^t$

$V'(3) = 20000 \ln\frac{3}{4} \cdot \left(\frac{3}{4}\right)^3 \simeq -2427.32$

AFTER 3 YEARS THE CAR IS DEPRECIATING AT A RATE OF

\$ 2427.32 PER YEAR.

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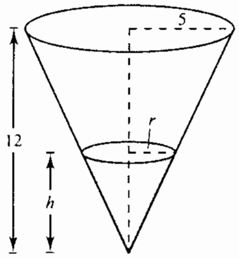
6. (*15 pts*) A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

ASKED FOR h' WHEN $V' = 10\ ft^3/min$ AND $h = 8\ ft$



$$\frac{r}{h} = \text{CONST.} = \frac{5}{12} \rightarrow r = \frac{5}{12}h \Big] \rightarrow$$

$$V = \frac{1}{3}\pi r^2 h$$

$$\rightarrow V = \frac{1}{3}\pi\left(\frac{5}{12}h\right)^2 h = \frac{25}{3\cdot 144}\pi h^2\cdot h \rightarrow$$

$V = \frac{1}{3}\pi r^2 h$

$$\rightarrow V = \frac{25\pi}{3\cdot 144}h^3 \rightarrow V' = \frac{25\pi}{3\cdot 144}3h^2 h' \rightarrow$$

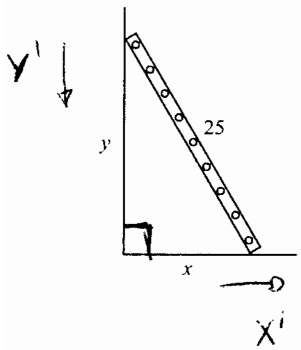
$$V' = \frac{d}{dt}[V], \quad h' = \frac{d}{dt}[h]$$

$$\rightarrow V' = \frac{25\pi}{144}h^2 h' \rightarrow h' = \frac{144 V'}{25\pi h^2}$$

AT THE GIVEN MOMENT, PLUG DATA: $h' = \dfrac{144\cdot 10}{25\pi\cdot 8^2} = \dfrac{9}{10\pi} \approx .29$

WHEN THE WATER IS AT 8 ft AND FLOWING AT 10 ft^3/min, THE WATER IS RISING AT A RATE OF ABOUT .29 ft PER MINUTE.

7. (*15 pts*) A ladder 25 feet long is leaning against the wall of a house (see scheme). The base of the ladder is pulled away at a rate $x' = 2$ feet per second. How fast is the top of the ladder moving down the wall when its base is 7 feet from the wall?



$x' = \frac{dx}{dt} = 2$ ft/sec. ASKED FOR Y' WHEN $X = 7$ ft.

$x^2 + y^2 = 25^2 \rightarrow \frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}\left[25^2\right] \rightarrow$

$\rightarrow 2xx' + 2yy' = 0$

WHEN $X = 7$ THEN $7^2 + Y^2 = 25^2 \rightarrow$

$\rightarrow y^2 = 25^2 - 7^2 = 24^2 \rightarrow Y = 24$

$Y > 0$

PLUG DATA

$\rightarrow 2 \cdot 7 \cdot 2 + 2 \cdot 24 \cdot y' = 0 \rightarrow 2 \, 24 \, y' = -28 \rightarrow$

$\rightarrow y' = -\frac{28}{2 \cdot 24} = -\frac{7}{12} \approx -.58$ ft/sec.

WHEN THE BASE IS 7 FEET FROM THE WALL, THE TOP OF THE LADDER IS FALLING AT A SPEED OF ABOUT .58 FEET PER SECOND.

8. (*15 pts*) A population of 500 bacteria is introduced into a culture and grows in number according to the model

$$P(t) = 500\left(1 + \frac{4t}{50+t^2}\right),$$

where t is measured in hours. Find the rate at which the population is growing ~~when~~ after two hours.

$$\text{RATE} = \frac{dP}{dt} = P' = 500 \cdot \frac{d}{dt}\left[1 + \frac{4t}{50+t^2}\right] = 500\,\frac{d}{dt}\left[\frac{4t}{50+t^2}\right] =$$

$$= 500 \cdot 4 \cdot \frac{d}{dt}\left[\frac{t}{50+t^2}\right] = 2000\,\frac{\frac{dt}{dt}\cdot(50+t^2) - t\cdot\frac{d}{dt}[50+t^2)}{(50+t^2)^2} =$$

$$= 2000\,\frac{50+t^2 - t\cdot 2t}{(50+t^2)^2} = \frac{(50-t^2)\,2000}{(50+t^2)^2}$$

$$P'(2) = \frac{(50-4)2000}{(50+4)^2} = \frac{23000}{729} \approx 31.55 \text{ BACTERIA PER HOUR.}$$

AFTER TWO HOURS, THE POPULATION IS GROWING AT A RATE OF ABOUT 32 BACTERIA PER HOUR.