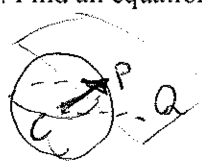
MAT 421 - Exam1 – Fall 2015 – Take Home Part

Instructor: Dr. Francesco Strazzullo **Name** KEY

Instructions. Complete 7 out of the following 10 exercises, according to the instructions. Each exercise is worth 10 points.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

Complete the following exercises.

1. Find an equation of the sphere with center $C = (1, -2, 1)$ that is tangent to the plane $2x - y + z = -3$.



TANGENCY $\Rightarrow$ "RADIUS VECTOR" $= \vec{CP} =$ "DIREC. VECT. OF PLANE" $\Rightarrow$
$\Rightarrow \vec{CP} = R\langle 2, -1, 1\rangle$, WHERE R IS THE RADIUS.

$R =$ "DISTANCE OF C FROM PLANE" $= \|\text{proj}_{\vec{n}}\vec{CQ}\|$, WHERE Q IS ANY POINT ON PLANE. TAKE $Q = (0, 0, -3) \Rightarrow \vec{CQ} = \langle -1, 2, -4\rangle \Rightarrow R = \dfrac{|\vec{CQ}\cdot\vec{n}|}{\|\vec{n}\|}$

$$= \frac{|2\cdot(-1) - 1\cdot 2 + 1(-4)|}{\sqrt{4+1+1}} = \frac{8}{\sqrt{6}} = \frac{4}{3}\sqrt{6} \Rightarrow R^2 = \frac{32}{3}$$

$$\boxed{\text{EQUATION OF SPHERE: } (x-1)^2 + (y+2)^2 + (z-1)^2 = \frac{32}{3}}$$

NOTE FORMULA: $R = \dfrac{|\langle x_0, y_0, z_0\rangle\cdot\vec{n} - d|}{\|\vec{n}\|} = \dfrac{|2\cdot 1 - 1\cdot(-2) + 1\cdot 1 - (-3)|}{\sqrt{6}} = \dfrac{8}{\sqrt{6}}$

2. Determine all values for a such that the vectors $\mathbf{x} = 3\mathbf{i} + a\mathbf{j} - 2\mathbf{k}$ and $\mathbf{y} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ will form a 30° angle.

$\vec{x}\cdot\vec{y} = \|\vec{x}\|\cdot\|\vec{y}\|\cdot\cos 30° \Rightarrow 3 + 2a - 2 = \sqrt{9 + a^2 + 4}\cdot\sqrt{1+4+1}\cdot\dfrac{\sqrt{3}}{2}$

$\Rightarrow 1 + 2a = \dfrac{3\sqrt{2}}{2}\sqrt{a^2 + 13} \Rightarrow 2(1+2a)^2 = \dfrac{9}{2}(a^2+13) \Rightarrow 2 + 8a + 8a^2 =$

$= 9a^2 + 117 \Rightarrow a^2 - 8a + 115 = 0 \Rightarrow a = 4 \pm 3i\sqrt{11}$, REJECTED BECAUSE a MUST BE REAL.

$\boxed{\text{NONE REAL NUMBER } a \text{ CAN BE FOUND}}$

3. Let $f(x, y) = \log_3(y^2 - x + 2)$.

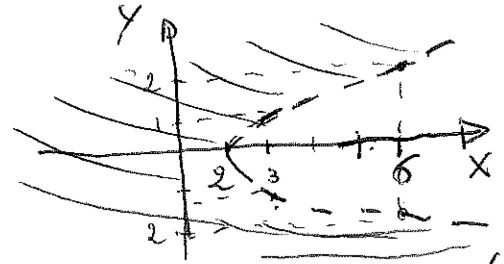
(a) Evaluate $f(2, 3)$.

(b) Find the domain of f and describe it graphically.

(c) Find the range of f.

(a) $f(2,3) = \log_3 9 = 2$; (b) $y^2 - x + 2 > 0 \Rightarrow x < y^2 + 2 \Rightarrow$

$\Rightarrow$ REGION TO THE LEFT OF PARABOLA $X = y^2 + 2$



(c) NOTE $f(x,y) = \log_3(g(x,y))$, WHERE $g(x,y) = y^2 - x + 2 \in (0, +\infty)$ ON OUR DOMAIN (b), THEN THE RANGE OF $f(x,y)$ IS THE SAME AS THAT OF THE STANDARD $\log_3 t$ FOR $t \in (0, +\infty)$, THAT IS THE RANGE OF $f(x,y)$ IS $(-\infty, \infty)$ (NOTE THE PARABOLIC CYLINDER $X = Y^2 + 2$ IS AN ASYMPTOTIC SURFACE TO $f(x,y)$)

4. Let $\mathbf{a}(t)$, $\mathbf{v}(t)$, and $\mathbf{r}(t)$ denote the acceleration, velocity, and position at time t of an object moving in the *Euclidean Space*. Find $\mathbf{r}(t)$, given that

$$\mathbf{a}(t) = \langle \cos 2t + t, \quad t^2, \quad e^{3t} \rangle, \mathbf{v}(0) = \langle -1, 0, 1 \rangle, \text{and } \mathbf{r}(0) = \langle 1, 1, 0 \rangle.$$

$$\vec{V}(t) = \int \vec{a}(t)\,dt = \langle \tfrac{1}{2}\sin(2t) + \tfrac{1}{2}t^2, \tfrac{t^3}{3}, \tfrac{1}{3}e^{3t} \rangle + \vec{C}$$

PLUG $\vec{V}(0) \Rightarrow \langle -1, 0, 1 \rangle = \langle 0, 0, \tfrac{1}{3} \rangle + \vec{C} \Rightarrow \vec{C} = \langle -1, 0, \tfrac{2}{3} \rangle$

THEN $\vec{V}(t) = \langle \tfrac{1}{2}(\sin(2t) + t^2) - 1, \tfrac{t^3}{3}, \tfrac{1}{3}(e^{3t} + 2) \rangle$

$$\vec{r}(t) = \int \vec{V}(t)\,dt = \langle -\tfrac{1}{4}\cos(2t) + \tfrac{1}{6}t^3 - t, \tfrac{t^4}{12}, \tfrac{1}{9}e^{3t} + \tfrac{2}{3}t \rangle + \vec{C}$$

PLUG $\vec{r}(0) \Rightarrow \langle 1, 1, 0 \rangle = \langle -\tfrac{1}{4}, 0, \tfrac{1}{9} \rangle + \vec{C} \Rightarrow \vec{C} = \langle \tfrac{5}{4}, 1, -\tfrac{1}{9} \rangle$

THEN $\vec{r}(t) = \langle -\tfrac{1}{4}\cos(2t) + \tfrac{1}{6}t^3 - t + \tfrac{5}{4}, \tfrac{t^4}{12} + 1, \tfrac{1}{9}e^{3t} + \tfrac{2}{3}t - \tfrac{1}{9} \rangle$

5. Write in cylindrical coordinates the parametric equations of curve with vector equation **r**(t) = 3 sin(2t) **i** + 3 cos(2t) **j** + t **k**.

CARTESIAN $\begin{cases} x = 3\sin(2t) \\ y = 3\cos(2t) \\ z = t \end{cases}$ CYLINDRICAL RELATIONS $\begin{cases} x^2 + y^2 = r^2 \\ \tan\theta = y/x \\ z = z \end{cases}$

THEN: $r^2 = (3\sin(2t))^2 + (3\cos(2t))^2 = 9(\sin^2(2t) + \cos^2(2t)) = 9 \Rightarrow$

$\Rightarrow r^2 = 9 \Rightarrow r = 3$

$\tan\theta = \dfrac{3\cos(2t)}{3\sin(2t)} = \cot(2t) \Rightarrow \theta = \dfrac{\pi}{2} - 2t$ (FUND. IDENTITIES FRONT PAGE 2)

$\begin{cases} r = 3 \\ \theta = \dfrac{\pi}{2} - 2t \\ z = t \end{cases}$

Complete one of the following three exercises.

6. Let **a** = 5**i** – 2**j** – 3**k** and **b** = **i** –**j** + 4**k**. Find an equation of the line parallel to **a** – **b** and passing through the tip of **a**.

$\vec{x} = \vec{a} + t(\vec{a} - \vec{b}) = \langle 5, -2, -3 \rangle + t\langle 4, -1, -7 \rangle$ (ENOUGH)

PARAM. EQ. $\begin{cases} x = 5 + 4t \\ y = -2 - t \\ z = -3 - 7t \end{cases}$ SYMM. EQ. $\dfrac{x-5}{4} = \dfrac{y+2}{-1} = \dfrac{z+3}{-7}$

7. Find parametric equations and symmetric equations of the line passing through the points $A = (2, -1, 0)$ and $B = (3, 2, -4)$.

$\vec{X} = \vec{a} + t\vec{AB} = \langle 2, -1, 0 \rangle + t \langle 1, 3, -4 \rangle$.

PAR. EQ. $\boxed{\begin{cases} x = 2 + t \\ y = -1 + 3t \\ z = -4t \end{cases}}$

SYMM. EQ: $\boxed{\dfrac{x-2}{1} = \dfrac{y+1}{3} = \dfrac{z}{-4}}$

8. Let L be the line given by $x = 1 - 3t$, $y = 2 + 2t$, and $z = 1 + t$, and P be the point $(5, 0, 2)$. Find parametric equations for the line through P which is parallel to L.

DIRECT. VECT OF L: $\langle -3, 2, 1 \rangle$

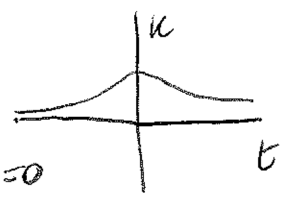
$\boxed{\begin{cases} x = 5 - 3t \\ y = 2t \\ z = 2 + t \end{cases}}$

Complete one of the following two exercises.

9. At what point does the curve $\mathbf{r}(t) = \langle t^2, 2\cos t, 2\sin t\rangle$ have minimum curvature? What is the minimum curvature?

$$\vec{r}\,'(t) = \langle 2t, -2\sin t, 2\cos t\rangle \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{4t^2 + 4\sin^2 t + 4\cos^2 t} =$$
$$= \sqrt{4t^2+4} = 2\sqrt{t^2+1} \quad ; \quad \vec{r}\,''(t) = 2\langle 1, -\cos t, -\sin t\rangle \Rightarrow$$

$$\kappa = \frac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \frac{\|4\langle -1, t\sin t + \cos t, -t\cos t + \sin t\rangle\|}{8\,(t^2+1)^{3/2}} =$$
$$= \frac{1}{2}(t^2+1)^{-3/2}\cdot\sqrt{1 + \underline{t^2\sin^2 t} + \underline{\cos^2 t} + 2t\sin t\cos t + \underline{t^2\cos^2 t} + \underline{\sin^2 t} - 2t\sin t\cos t}$$
$$= \frac{1}{2}(t^2+1)^{-3/2}\cdot\sqrt{1+t^2+1} = \frac{1}{2}(t^2+1)^{-3/2}(t^2+2)^{1/2}$$



$\boxed{\text{THERE IS NONE MINIMUM}}$, THERE IS A MAX FOR $t=0$

WITH MAX CURVATURE $\kappa(0) = \sqrt{2}/2$.

NOTE: $\kappa'(t) = -\frac{1}{2}t(2t^2+5)(t^4+3t^2+2)^{\frac{1}{2}}(t^2+2)^{-1}(t^2+1)^{-3}$

10. Find the center of the osculating circle of the curve $\mathbf{r}(t) = \langle e^t, 2t^3, e^{-t}\rangle$ at $P = (1,0,1)$.

P ON TRAJECTORY OF $\vec{r}(t)$ FOR $\begin{cases} e^t = 1 \\ 2t^3 = 0 \\ e^{-t} = 1 \end{cases} \to t = 0$ ✓

WE NEED TO FIND THE RADIUS VECTOR $\vec{PC} = R\vec{N}(0) = \frac{1}{\kappa(0)}\vec{N}(0)$.

$$\vec{r}\,'(t) = \langle e^t, 6t^2, -e^{-t}\rangle \Rightarrow \vec{T}(t) = \frac{1}{\sqrt{36t^4 + e^{2t} + e^{-2t}}}\vec{r}\,'(t)$$

$$\vec{T}\,'(0) = \frac{\sqrt{2}}{2}\langle 1, 0, 1\rangle \Rightarrow \|\vec{T}\,'(0)\| = 1 \Rightarrow \vec{N}(0) = \vec{T}\,'(0)$$

USING CLB $\quad \kappa(0) = \frac{\|\vec{T}\,'(0)\|}{\|\vec{r}\,'(0)\|} = \frac{1}{\|\langle 1, 0, -1\rangle\|} = \frac{\sqrt{2}}{2}$

THEN $\vec{PC} = \langle 1, 0, 1\rangle \Rightarrow \langle x_c - 1, y_c, z_c - 1\rangle = \langle 1, 0, 1\rangle$

$$\Rightarrow \begin{cases} x_c - 1 = 1 \Rightarrow x_c = 2 \\ y_c = 0 \\ z_c - 1 = 1 \Rightarrow z_c = 2 \end{cases} \Rightarrow \boxed{C = (2, 0, 2)}$$

MAT 421 - Exam1 – Fall 2015 – In Class Part

Instructor: Dr. Francesco Strazzullo **Name** KEY

Instructions. Complete 3 out of the following 5 exercises, according to the instructions. Each exercise is worth 10 points.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

Complete one of the following two exercises.
1. Find parametric equations of the tangent line to the curve $\mathbf{r}(t) = \langle e^{2t}, \sqrt[3]{t}, e^{-2t} \rangle$ at $P = (1,0,1)$.

DIRECT. VECT $= \vec{r}\,'(t) = \langle 2e^{2t}, \frac{1}{3}t^{-2/3}, -2e^{-2t} \rangle$

AT $P = (1,0,1)$: $\begin{cases} e^{2t} = 1 \\ \sqrt[3]{t} = 0 \Rightarrow t = 0 \\ e^{-2t} = 1 \end{cases}$ $\Rightarrow$

$\Rightarrow$ DIR. VECT AT P IS $\vec{r}\,'(0)$ BUT THE SECOND COMPONENT IS NOT DEFINED FOR $t=0$, THEN THE TANGENT LINE

2. Find the directional vector of the line which is the intersection of the planes $2x + 3y - z = -2$ and $x - y - 3z = 1$.

$\pi_i: a_i x + b_i y + c_i z = d_i$ HAS DIRECT. VECT $\vec{n}_i = \langle a_i, b_i, c_i \rangle$

INTERSECTION LINE HAS DIR. VECT $= \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 1 & -1 & -3 \end{vmatrix}$

$= \boxed{\langle -10, 5, -5 \rangle}$

Complete two of the following three exercises.

3. Find the unit tangent and the unit normal to the trajectory of the vector function $\mathbf{r}(t) = \langle t, -t^2, 0\rangle$ at $P = (1, -1, 0)$.

AT P : $\begin{cases} t = 1 \\ -t^2 = -1 \\ 0 = 0 \end{cases} \Rightarrow t = 1.$ LOOKING FOR $\vec{T}(1)$ AND $\vec{N}(1)$

$\vec{r}'(t) = \langle 1, -2t, 0 \rangle \Rightarrow \vec{T}(t) = \frac{1}{\sqrt{1+4t^2}} \vec{r}'(t)$, THEN

$$\boxed{\vec{T}(1) = \frac{1}{\sqrt{5}} \langle 1, -2, 0 \rangle}$$

$\vec{T}'(1) \underset{\text{GGB}}{=} -2\frac{\sqrt{5}}{25} \langle 2, 1, 0 \rangle \Rightarrow \|\vec{T}'(1)\| = \frac{2}{5}$, THEN

$$\boxed{\vec{N}(1) = \frac{\vec{T}'(1)}{\|\vec{T}'(1)\|} = -\frac{1}{\sqrt{5}} \langle 2, 1, 0 \rangle}$$

4. Let $\mathbf{a}(t)$, $\mathbf{v}(t)$, and $\mathbf{r}(t)$ denote the acceleration, velocity, and position at time t of an object moving in the xz-plane. Find $\mathbf{r}(t)$, given that

$$\mathbf{a}(t) = \langle \sin 2t - 3t, 0, \quad t^3 \ \rangle, \mathbf{v}(0) = \langle 2, 0, -1 \rangle, \text{and } \mathbf{r}(0) = \langle -1, 0, 2 \rangle.$$

$\vec{V}(t) = \int \vec{a}(t)\,dt = \langle -\frac{1}{2}\cos(2t) - \frac{3}{2}t^2, 0, \frac{t^4}{4} \rangle + \vec{C}$, PLUG $\vec{V}(0) \Rightarrow$

$\Rightarrow \langle 2, 0, -1 \rangle = \langle -\frac{1}{2}, 0, 0 \rangle + \vec{C} \Rightarrow \vec{C} = \langle \frac{5}{2}, 0, -1 \rangle \Rightarrow$

$\Rightarrow \vec{V}(t) = \langle -\frac{1}{2}\cos(2t) - \frac{3}{2}t^2 + \frac{5}{2}, 0, \frac{t^4}{4} - 1 \rangle$, $\vec{r}(t) = \int \vec{V}(t)\,dt \Rightarrow$

$\Rightarrow \vec{r}(t) = \langle -\frac{1}{4}\sin(2t) - \frac{1}{2}t^3 + \frac{5}{2}t, 0, \frac{t^5}{20} - t \rangle + \vec{C}$, PLUG $\vec{r}(0) \Rightarrow$

$\Rightarrow \langle -1, 0, 2 \rangle = \langle 0, 0, 0 \rangle + \vec{C} \Rightarrow \vec{C} = \langle -1, 0, 2 \rangle$ THEN

$\vec{r}(t) = \langle -\frac{1}{4}\sin(2t) - \frac{1}{2}t^3 + \frac{5}{2}t - 1, 0, \frac{1}{20}t^5 - t + 2 \rangle$

5. Find a parametric representation for the surface consisting of that part of the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{36} = 1$$

that lies between the planes $z = -2$ and $z = 4$.

SET $z = s$, $-2 \le s \le 4$, THEN

$\frac{x^2}{4} + \frac{y^2}{9} + \frac{s^2}{36} = 1 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1 - \frac{s^2}{36}$ IS AN ELLIPSE

FOR EACH VALUE OF s, BUT NOT IN STANDARD FORM, $-2 \le s \le 4 \Rightarrow$

$s \ne \pm 6$, $0 \ne 1 - \frac{s^2}{36} = \frac{36 - s^2}{36} > 0$. THEN STANDARD FOR. OF ELLIPSE IS

$$\frac{x^2}{\frac{36-s^2}{9}} + \frac{y^2}{\frac{36-s^2}{4}} = 1 \Rightarrow \begin{cases} x = \frac{\sqrt{36-s^2}}{3}\cos t \\ y = \frac{\sqrt{36-s^2}}{2}\sin t \end{cases} \quad 0 \le t < 2\pi$$

ANSWER:

$$\begin{cases} x = \frac{1}{3}\sqrt{36-s^2}\cos t \\ y = \frac{1}{2}\sqrt{36-s^2}\sin t \\ z = s \end{cases}, \quad 0 \le t < 2\pi, \ -2 \le s \le 4$$