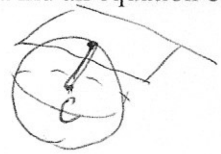
MAT 421 - Exam1 – Fall 2017 – Take Home Part

Instructor: Dr. Francesco Strazzullo **Name** KEY

I certify that I did not receive third party help in *completing* this test (sign)____________________

Instructions. This is an open book test. Each exercise is worth 10 points. When approximating, use four decimal places. You cannot use a CAS to justify your answers, but only to perform computations.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Find an equation of the sphere with center $C = (2,-1,-2)$ that is tangent to the plane $3x + y - 2z = 3$.



$R = dist(C, \pi) = \|\langle a,b,c\rangle\|^{-1} \cdot |a x_0 + b y_0 + c z_0 - d| \Rightarrow$

π-PLANE: $3x+y-2z=3$

$\Rightarrow R = (3^2+1^2+(-2)^2)^{-\frac{1}{2}} \cdot |3\cdot 2 + 1(-1) - 2(-2) - 3| = \frac{6}{\sqrt{14}} \Rightarrow R^2 = \frac{18}{7}$

EQ.: $(x-2)^2 + (y+1)^2 + (z+2) = \frac{18}{7}$

2. Determine all values for a such that the vectors $\mathbf{x} = a\mathbf{i} + 2a\mathbf{j} - 5\mathbf{k}$ and $\mathbf{y} = 3a\mathbf{i} - a\mathbf{j} + 14\mathbf{k}$ are orthogonal.

ORTHO $\Rightarrow 0 = \vec{x} \cdot \vec{y} = \langle a, 2a, -5\rangle \cdot \langle 3a, -a, 14\rangle = 3a^2 - 2a^2 - 70 \Rightarrow$

$\Rightarrow a^2 - 70 = 0 \Rightarrow a = \pm\sqrt{70}$

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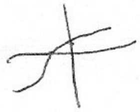
3. Let $f(x, y) = \sqrt{x - y^3 + 1} + 3y$.

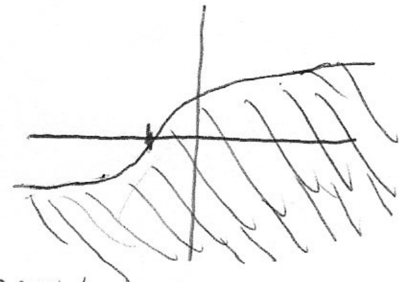
(a) Evaluate $f(4, 1)$. $= \sqrt{4 - 1 + 1} + 3(1) = 5$

(b) Find the domain of f (algebraically) and describe it graphically (draw a graph of this domain).

(c) Find the range of f.

(b) $x - y^3 + 1 \geq 0 \Rightarrow x \geq y^3 - 1$

BL: $x = y^3 - 1$  TEST $(0,0): 0 \geq -1$ ✓



(c) THE $3y$ TERM IS UNBOUNDED AND ON THE DOMAIN THE $\sqrt{x - y^3 + 1}$ TERM IS NON-NEGATIVE. THEREFORE $f(x)$ IS UNBOUNDED

$\text{Img}(f) = (-\infty, \infty)$

4. Let **a** = 3**i** + 4**j** – 2**k** and **b** = **i** – **j** + 3**k**. Find an equation of the line parallel to **a** + **b** and passing through the tip of **b**.

DIRECTIONAL VECTOR : $\vec{u} = \vec{a} + \vec{b} = \langle 4, 3, 1 \rangle$

ONE POINT = "TIP OF $\vec{b}$" $= (1, -1, 3)$ $\Big] \Rightarrow$

$\Rightarrow \vec{r} = \vec{b} + t\vec{u} = \langle 1, -1, 3 \rangle + \langle 4t, 3t, t \rangle = \langle 4t+1, 3t-1, t+3 \rangle$

ALTERNATIVE $\begin{cases} x = 4t + 1 \\ y = 3t - 1 \\ z = t + 3 \end{cases}$ OR $\dfrac{x-1}{4} = \dfrac{y+1}{3} = z - 3$

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5. Write in Cartesian coordinates the parametric equations of the curve with ***cylindrical vector equation*** $\mathbf{r}(t) = \langle t^2, \frac{\pi}{4}, t\rangle$.

$$\vec{r} \equiv \begin{cases} r = t^2 \\ \theta = \frac{\pi}{4} \\ z = t \end{cases} \quad \text{BUT} \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \text{, THEN} \quad \begin{cases} x = t^2/\sqrt{2} \\ y = t^2/\sqrt{2} \\ z = t \end{cases} .$$

$$\vec{r}(t) = \left\langle \frac{t^2}{\sqrt{2}}, \frac{t^2}{\sqrt{2}}, t \right\rangle$$

NOTE THE IMPLICIT FORM $\begin{cases} y = x \\ x = \frac{z^2}{\sqrt{2}} \end{cases}$

6. Let $\mathbf{a}(t)$, $\mathbf{v}(t)$, and $\mathbf{r}(t)$ denote the acceleration, velocity, and position at time t of an object moving in the *Euclidean Space*. Find $\mathbf{r}(t)$, given that
$\mathbf{a}(t) = \langle 2t + \cos(\pi t), t, 0\rangle$, $\mathbf{v}(1) = \langle -1, 0, 1\rangle$, and $\mathbf{r}(1) = \langle 1, -1, 1\rangle$.

$$V(t) = \int a(t)\,dt = \left\langle t^2 + \frac{1}{\pi}\sin(\pi t) + c_1, \frac{t^2}{2} + c_2, c_3 \right\rangle \quad \text{PLUG CONDITION}$$

$$\langle -1, 0, 1\rangle = V(1) = \left\langle 1^2 + \frac{1}{\pi}\sin(\pi) + c_1, \frac{1}{2} + c_2, c_3 \right\rangle = \left\langle 1 + c_1, \frac{1}{2} + c_2, c_3 \right\rangle \Rightarrow$$

$$\Rightarrow \langle c_1, c_2, c_3\rangle = \left\langle -1-1, -\frac{1}{2}, 1 \right\rangle = \left\langle -2, -\frac{1}{2}, 1 \right\rangle \Rightarrow$$

$$\Rightarrow v(t) = \left\langle t^2 + \frac{1}{\pi}\sin(\pi t) - 2, \frac{t^2}{2} - \frac{1}{2}, 1 \right\rangle \Rightarrow r(t) = \int v(t)\,dt =$$

$$= \left\langle \frac{t^3}{3} - \frac{1}{\pi^2}\cos(\pi t) - 2t + c_1, \frac{t^3}{6} - \frac{1}{2}t + c_2, t + c_3 \right\rangle \quad \text{PLUG CONDITION}$$

$$\langle 1, -1, 1\rangle = \vec{r}(1) = \left\langle \frac{1}{3} + \frac{1}{\pi^2} - 2 + c_1, -\frac{1}{3} + c_2, 1 + c_3 \right\rangle \Rightarrow$$

$$\Rightarrow \langle c_1, c_2, c_3\rangle = \left\langle \frac{8}{3} - \frac{1}{\pi^2}, -\frac{2}{3}, 0 \right\rangle \Rightarrow$$

$$\Rightarrow \vec{r}(t) = \left\langle \frac{t^3}{3} - 2t + \frac{8}{3} - \frac{1}{\pi^2} - \frac{1}{\pi^2}\cos(\pi t), \frac{t^3}{6} - \frac{1}{2}t - \frac{2}{3}, t \right\rangle$$

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7. Find parametric equations and symmetric equations of the line passing through the points $A = (1, 2, -1)$ and $B = (-1, 3, -2)$.

DIRECTIONAL VECTOR $= \vec{AB} = B - A = \langle -2, 1, -1 \rangle$ OR $\langle 2, -1, 1 \rangle$

$$\begin{cases} x = 1 + 2t \\ y = 2 - t \\ z = -1 + t \end{cases}$$ PARAMETRIC

SYMMETRIC: $\frac{x-1}{2} = \frac{y-2}{-1} = z+1$

8. Let L be the line given by $x = -1 + 3t$, $y = 1 + 2t$, and $z = 3 - t$, and P be the point $(1, 0, -1)$. Find parametric equations for the line through P which is parallel to L.

DIRECTIONAL VECTOR $= \langle 3, 2, -1 \rangle$ THEN

$\vec{r}(t) = \langle 1 + 3t, 2t, -1 - t \rangle$

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9. (**Main Section**) Using Calculus (not graphs), find a point where the curve $\mathbf{r}(t) = \langle 2t, 3t^2, 5-3t\rangle$ has maximum curvature. What is the maximum curvature?

$\vec{r}'(t) = \langle 2, 6t, -3 \rangle \Rightarrow \|\vec{r}'(t)\| = \sqrt{4+36t^2+9} = \sqrt{13+36t^2}$

$\vec{r}''(t) = \langle 0, 6, 0 \rangle \Rightarrow r' \times r'' = \begin{vmatrix} 2 & 6t & -3 \\ 0 & 6 & 0 \end{vmatrix} = \langle +18, 0, 12 \rangle = 6\langle 3, 0, 2 \rangle$

$$K(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{6\sqrt{13}}{\left(\sqrt{13+36t^2}\right)^3}$$

BECAUSE $K(t)$ HAS A CONSTANT NUMERATOR AND IT IS POSITIVE, THEN THE MAXIMUM IS ACHIEVED WHEN THE DENOMINATOR HAS A MINIMUM.

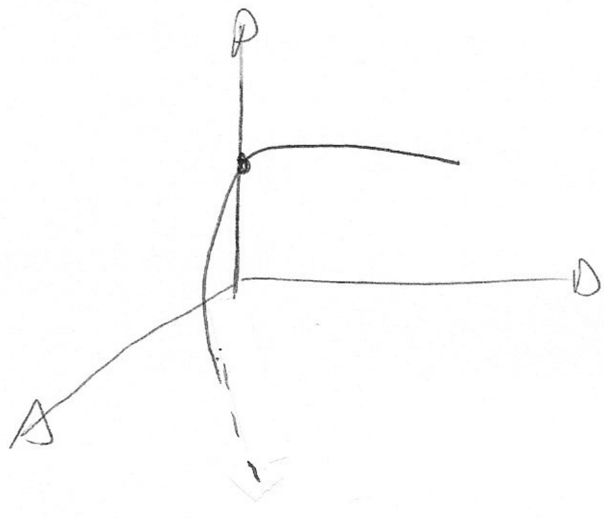
$f(t) = (13+36t^2)^{3/2}$

$f'(t) = 36t \cdot \frac{3}{2}(13+36t^2)^{\frac{1}{2}} = 0 \Rightarrow$ THE SIGN OF $f'(t)$ IS THE SAME AS THAT OF t.

t	-1	0	$+$
$f'(t)$	$-$		$+$

THEN $f(t)$ HAS MINIMUM AT $t = 0$, WHERE $K(t)$ HAS ITS MAXIMUM

$K(0) = \frac{6}{13}$, AT $\vec{r}(0) = \langle 0, 0, 5 \rangle$



10. **(Honor Section).** Find the center of the osculating circle of the curve $\mathbf{r}(t) = \langle 2t, 3t^2, 5-3t\rangle$ at $P = (2,3,2)$.

FROM #9: $r(t) = P \equiv \begin{cases} 2t = 2 \Rightarrow t = 1 \\ 3t^2 = 3 \quad \checkmark \\ 5 - 3t = 2 \quad \checkmark \end{cases}$

$\vec{r}\,'(t) = \langle 2, 6t, -3\rangle \Rightarrow \vec{T}(t) = \dfrac{1}{\sqrt{13+36t^2}}\langle 2, 6t, -3\rangle$

$\vec{T}\,'(t) = 36(2t)\left(-\frac{1}{2}\right)(13+36t^2)^{-\frac{3}{2}}\langle 2, 6t, -3\rangle + (13+36t^2)^{-\frac{1}{2}}\langle 0, 6, 0\rangle$

$\vec{T}\,'(1) = \dfrac{-36}{7^3}\langle 2, 6, -3\rangle + \dfrac{1}{7}\langle 0, 6, 0\rangle$

$= \dfrac{1}{7}\left\langle -\dfrac{72}{49}, \dfrac{6(49-36)}{49}, \dfrac{108}{49}\right\rangle =$

$= \dfrac{6}{7^3}\langle -12, 13, 18\rangle \Rightarrow \|\vec{T}\,'(1)\| = \dfrac{6}{7^3}\sqrt{637} = \dfrac{6}{7^2}\sqrt{13}$

$\vec{N}(1) = \dfrac{\vec{T}\,'(1)}{\|\vec{T}\,'(1)\|} = \dfrac{1}{7\sqrt{13}}\langle -12, 13, 18\rangle$

$\boxed{\#9} \Rightarrow \kappa(1) = \dfrac{6\sqrt{13}}{7^3} \Rightarrow R = \dfrac{1}{\kappa} = \dfrac{7^3}{6\sqrt{13}}$ $\Bigg] \Rightarrow$

$\Rightarrow \vec{PC} = R\vec{N}$ IS $\langle x_c - 2, y_c - 3, z_c - 2\rangle = \dfrac{7^2}{6\cdot 13}\langle -12, 13, 18\rangle \Rightarrow$

$\Rightarrow \begin{cases} x_c = -\dfrac{49\cdot 2}{13} + 2 = -\dfrac{72}{13} \\ y_c = \dfrac{49}{6} + 3 = 67/6 \\ z_c = \dfrac{49\cdot 3}{13} + 2 = 173/13 \end{cases} \Rightarrow C = \left(-\dfrac{72}{13}, \dfrac{67}{6}, \dfrac{173}{13}\right)$

EXTRA:

DIRECTION OF OSCULATING PLANE: $\vec{T}(1) \times \vec{N}(1) \parallel \langle 2, 6, -3\rangle \times \langle -12, 13, 18\rangle =$

$= \begin{vmatrix} 2 & 6 & -3 \\ -12 & 13 & 18 \end{vmatrix} = \langle 147, 0, 98\rangle = 49\langle 3, 0, 2\rangle \Rightarrow \langle a, b, c\rangle = \langle 3, 0, 2\rangle$

OSCULATING PLANE: $3x + 2z = \langle 3, 0, 2\rangle \cdot \langle 2, 3, 2\rangle = 7$

OSC. CIRCLE $\equiv \begin{cases} \left(x + \frac{72}{13}\right)^2 + \left(y - \frac{67}{6}\right)^2 + \left(z - \frac{173}{13}\right)^2 = \dfrac{7^6}{36\cdot 13} \\ 3x + 2z = 7 \end{cases}$