MAT 421 – Exam4 – Fall 2015

Instructor: Dr. Francesco Strazzullo **Name** KEY

Instructions. Each exercise is worth 10 points.

SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Find a function of $f(x,y,z)$ such that $\nabla f = \langle 2xy - 3z,\ x^2 - 1, 1 - 3x\rangle$. $= \langle P, Q, R \rangle$

$P = f_x \Rightarrow f = \int P\,dx = x^2y - 3zx + g(z,y)$

$Q = f_y \Rightarrow x^2 - 1 = x^2 + g_y \Rightarrow g_y = -1 \Rightarrow g = \int g_y\,dy = -y + h(z)$

$\rightarrow R = f_z \Rightarrow 1 - 3x = -3x + h_z \Rightarrow h_z = 1 \Rightarrow h = z + C$

THEN $f(x,y,z) = x^2y - 3zx - y + z + C$

2. Find the work done by the force $\boldsymbol{F} = \langle \cos x, y^2\rangle$ in moving an object from (1, 1) to (7, −5) along the path $\boldsymbol{C}$ given by $\boldsymbol{r}(t) = \langle 3t - 2,\ 4 - 3t\rangle$.

$(1,1) = \vec{r}(t) \Rightarrow \begin{cases} 3t - 2 = 1 \Rightarrow \boxed{t=1} \\ 4 - 3t = 1 \ ✓ \end{cases}$; $(7,-5) = \vec{r}(t) \Rightarrow \begin{cases} 3t - 2 = 7 \Rightarrow \boxed{t=3} \\ 4 - 3t = -5 \ ✓ \end{cases}$

NOTE: $F = \langle P, Q\rangle$ IS SUCH THAT $Q_x = 0 = P_y \Rightarrow F$ IS CONSERVATIVE

WITH POTENTIAL f: $f_x = P \Rightarrow f = \sin x + g(y) \Rightarrow y^2 = Q = f_y =$

$= g' \Rightarrow g = \frac{1}{3}y^3 \Rightarrow f = \sin x + \frac{1}{3}y^3 + C$.

$W = \int_C \vec{F} \cdot d\vec{r} = f(x(3), y(3)) - f(x(1), y(1)) =$

$= \sin 7 + \frac{1}{3}(-5)^3 - \sin 1 - \frac{1}{3}1^3 = \sin 7 - \sin 1 - 42$

≈ -42.18 J.

3. Evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where $\boldsymbol{F} = \langle z^2,\ x,\ x-y \rangle$ and the curve C is given by the vector function $\boldsymbol{r}(t) = \langle t^3,\ t^2,\ t \rangle$ for $-1 \le t \le 1$. NOTE: $P_z = 0 \neq R_x = 1 \Rightarrow \vec{F}$ IS NOT CONSERVATIVE.

$$\vec{r}\,'(t) = \langle 3t^2, 2t, 1 \rangle ; \quad \vec{F}(\vec{r}(t)) = \langle t^2, t^3, t^3 - t^2 \rangle \Rightarrow \int_C \vec{F} \cdot d\vec{r} =$$

$$= \int_{-1}^{1} 3t^4 + 2t^4 + t^3 - t^2 \, dt = \int_{-1}^{1} 5t^4 + t^3 - t^2 \, dt$$

$$= \left[t^5 + \frac{1}{4}t^4 - \frac{1}{3}t^3 \right]_{-1}^{1} = 1 + \frac{1}{4} - \frac{1}{3} - (-1) - \frac{1}{4} + \frac{1}{3}(-1) = \frac{4}{3}$$

4. Determine whether $\boldsymbol{F} = \langle 1 + y\cos x, -1 + \sin x \rangle$ is a conservative vector field. If it is, find a function $f(x,y)$ such that $\boldsymbol{F} = \nabla f$.

USE CURL $\vec{F} = \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle \stackrel{?}{=} 0$ IN THIS CASE THINK ABOUT $\vec{F}$ AS A PROJECTION TO THE XY-PLANE FOR $\langle P, Q, 0 \rangle$

$Q_x = \cos x = P_y$ ✓.

$P = f_x = 1 + y\cos x \Rightarrow f = \int 1 + y\cos x \, dx = x + y\sin x + g(y) \Rightarrow$

$\Rightarrow [Q = f_y] \Rightarrow -1 + \sin x = \sin x + g' \Rightarrow g = \int -dy = -y + c$

THEN

$f(x,y) = x + y\sin x - y + c$

5. Find a function $f(x, y)$ such that $\boldsymbol{F} = \nabla f$ and use it to evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where $\boldsymbol{F} = \langle x^2,\ 3y \rangle$ and C is the arc of the curve $y^2 - 2y + x = 3$ from (3, 0) to (4, 1).

$P_y = 0 = Q_x$ ✓ $\Rightarrow f_x = P = x^2 \Rightarrow f = \int x^2\,dx = \frac{1}{3}x^3 + g(y) \Rightarrow$

$\Rightarrow 3y = Q = f_y = g' \Rightarrow g = \int 3y\,dy = \frac{3}{2}y^2 + C$. THEN WE

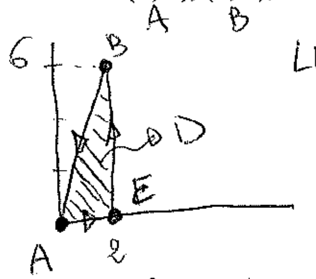
CAN USE $f(x,y) = \frac{1}{3}x^3 + \frac{3}{2}y^2$

$$\int_C \vec{F}\cdot d\vec{r} = f(4,1) - f(3,0) = \frac{1}{3}4^3 + \frac{3}{2} - \frac{1}{3}3^3 - \frac{3}{2}\cdot 0$$

$$= \frac{83}{6} \approx 13.8\bar{3}$$

6. Use Green's Theorem to evaluate the line integral of $\boldsymbol{F} = \langle y^2 \cos x, x^2 + 2y \sin x \rangle$ around the triangle with vertices (0, 0), (2, 6), and (2, 0), with counterclockwise orientation.

A B E



LINE AB : $Y = \frac{6}{2}X = 3X$. $\Rightarrow$ TRIANGLE $D = \{0 \le x \le 2,\ 0 \le y \le 3x\}$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D (Q_x - P_y)\,dA = \int_0^2 \int_0^{3x} 2x + 2y\cos x - 2y\cos x\,dy\,dx$$

$$= \int_0^2 \int_0^{3x} 2x\,dy\,dx = \int_0^2 2x\cdot 3x\,dx = \left[2x^3\right]_0^2 = 16$$

7. Find (a) the curl and (b) the divergence of the vector field $\boldsymbol{F} = \langle y^2 + \cos x, x^2 + \sin y,\ z^2 - x\rangle$.

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle$$

$$= \langle 0 - 0, -(-1-0), 2x - 2y \rangle = \langle 0, 1, 2x - 2y \rangle$$

$$\text{div } \vec{F} = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \vec{F} = -\sin x + \cos y + 2z$$

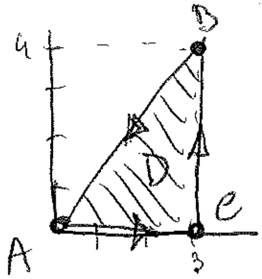
MAT 421 – Exam4 – Part 2/2– Fall 2015

Instructor: Dr. Francesco Strazzullo **Name** KEY

Instructions. Each exercise is worth 15 points.

SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Use Green's Theorem to evaluate the line integral of $\boldsymbol{F} = \langle e^{x+y}, y^2 - x\cos y\rangle$ around the triangle with vertices (0, 0), (3, 4), and (3, 0), with counterclockwise orientation.



LINE AB: $y = \frac{4}{3}x$; $D = \{0 \le x \le 3,\ 0 \le y \le \frac{4}{3}x\}$

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \iint_D Q_x - P_y \, dA = \int_0^3 \int_0^{4/3 x} -e^{x+y} - \cos y \, dy\, dx$$

$$= \int_0^3 \left[-e^{x+y} - \sin y\right]_0^{4/3 x} dx = \int_0^3 -e^{\frac{7}{3}x} - \sin\left(\tfrac{4}{3}x\right) + e^x \, dx$$

$$= \left[-\frac{3}{7}e^{\frac{7}{3}x} + \frac{3}{4}\cos\left(\frac{4}{3}x\right) + e^x\right]_0^3$$

$$= -\frac{3}{7}e^7 + \frac{3}{4}\cos 4 + e^3 + \frac{3}{7} - \frac{3}{4} - 1$$

$$= e^3 - \frac{3}{7}e^7 + \frac{3}{4}\cos 4 - \frac{37}{28} \approx -451.7118$$

2. Find (a) the curl and (b) the divergence of the vector field $\boldsymbol{F} = \langle y + e^{3x}, x + 3\sin z, \ln(xz)\rangle$.

$$\text{curl } \vec{F} = \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle$$

$$= \langle 0 - 3\cos z, -\left(\frac{1}{x} - 0\right), 1 - 1 \rangle$$

$$= \langle -3\cos z, -\frac{1}{x}, 0 \rangle$$

$$\text{div } \vec{F} = P_x + Q_y + R_z = 3e^{3x} + 0 + \frac{1}{z} = 3e^{3x} + \frac{1}{z}$$