MAT 320 – Exam3 – Fall 2015

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I certify that I did not receive third party help in *completing* this test (sign) ____________

Instructions. Technology is allowed on this exam. Each problem is worth 10 points, except numbers 7 and 8. If you use formulas or properties from our book, make a reference. When using technology describe which commands (or keys typed) you used or print out and attach your worksheet.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1) Let $H = \{a + bx^3 + cx^4 \in P_4 : b = a - c\}$ be a subset of the real vector space of polynomials of degree at most 4. If H is a subspace of P_4 then provide one of its bases, otherwise show which property of subspaces H does not satisfy.

$\vec{p} \in H \Leftrightarrow \vec{p} = a + (a-c)x^3 + cx^4 = a + ax^3 - cx^3 + cx^4 \Leftrightarrow$

$\vec{p} = a(1+x^3) + c(-x^3+x^4) \Leftrightarrow \vec{p} \in \text{Span}\{1+x^3, -x^3+x^4\}$

THEREFORE $H = \text{Span}\{1+x^3, -x^3+x^4\}$ IS A VECT. SUBSPACE.

BECAUSE $1+x^3 \neq t \cdot (-x^3+x^4)$ FOR ALL REAL NUMBERS t,

THEN $\{1+x^3, -x^3+x^4\}$ IS LINEARLY INDEPENDENT, THUS

A BASIS FOR H.

2) Let $\mathcal{C} = \left\{C_1 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix},\ C_2 = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, C_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right\}$ be a subset of the real vector space of the 2-by-2 matrices $M_{2,2}$.

(a) Use the standard basis $\mathcal{E} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ to prove that $\mathcal{C}$ is linearly independent (Hint: for each $C_i \in \mathcal{C}$ consider the vector of components (or coordinate vector) $[C_i]_{\mathcal{E}}$.)

(b) Use part (a) to extend $\mathcal{C}$ to a basis for $M_{2,2}$.

(a) $[C_i]_{\mathcal{E}} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \Leftrightarrow C_i = \sum_{i=1}^{4} t_i E_i = t_1 E_{11} + t_2 E_{21} + t_3 E_{12} + t_4 E_{22} = \begin{bmatrix} t_1 & t_3 \\ t_2 & t_4 \end{bmatrix}$

BY THE "UNIQUE REPRESENTATION THEOREM" $\mathcal{C}$ IS LIN. INDEP. IF AND ONLY IF $\{[C_1]_{\mathcal{E}}, [C_2]_{\mathcal{E}}, [C_3]_{\mathcal{E}}\}$ IS LIN. INDEP. THIS IS EQUIVALENT TO SAY THAT RANK$\left([[C_1]_{\mathcal{E}}\ [C_2]_{\mathcal{E}}\ [C_3]_{\mathcal{E}}]\right) = 3$

$$[\mathcal{C}]_{\mathcal{E}} = \left[[C_1]_{\mathcal{E}}\ [C_2]_{\mathcal{E}}\ [C_3]_{\mathcal{E}}\right] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ THEN

RANK$([\mathcal{C}]_{\mathcal{E}}) = 3$ AND $\mathcal{C}$ IS LIN. INDEP.

(b) ORLATE RREF$([\mathcal{C}]_{\mathcal{E}})$ WITH "MISSING" PIVOTAL $\vec{e}_i$: IN OUR CASE $\vec{e}_4 = [E_{22}]_{\mathcal{E}}$. THEN $\{C_1, C_2, C_3, E_{22}\}$ IS A BASIS.

3) Consider two bases $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$ of $\mathbb{R}^3$.

(a) Find $P_{\mathcal{C}}^{\mathcal{B}}$, the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

(b) Compute $[\boldsymbol{x}]_{\mathcal{B}}$ for $\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

(c) Use part (a) and (b) to compute $[\boldsymbol{x}]_{\mathcal{C}}$.

(a) $P_{\mathcal{C}}^{\mathcal{B}} = \left[[\vec{b}_1]_{\mathcal{C}} \cdots [\vec{b}_n]_{\mathcal{C}} \right]$, $P_{\mathcal{B}}^{\mathcal{C}} = \left(P_{\mathcal{C}}^{\mathcal{B}}\right)^{-1}$, $[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{\mathcal{B}} [\vec{x}]_{\mathcal{B}}$

IF $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ IS THE STANDARD BASIS IN $\mathbb{R}^3$, THEN

$P_{\mathcal{E}}^{\mathcal{B}} = [\mathcal{B}] = [\vec{b}_1 \cdots \vec{b}_n] = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ AND $[\mathcal{C}] = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$P_{\mathcal{C}}^{\mathcal{B}} \overset{*}{=} P_{\mathcal{C}}^{\mathcal{E}} \cdot P_{\mathcal{E}}^{\mathcal{B}} = [\mathcal{C}]^{-1} \cdot [\mathcal{B}] \underset{\text{TECHNOLOGY}}{=} \begin{bmatrix} 11/3 & -3 & 1/3 \\ -1 & 1 & 0 \\ 1/3 & 1 & -1/3 \end{bmatrix}$$

* BECAUSE $[\vec{b}_j]_{\mathcal{C}} = P_{\mathcal{C}}^{\mathcal{E}} [\vec{b}_j]_{\mathcal{E}} \Rightarrow$ THE j-TH COLUMN OF $P_{\mathcal{C}}^{\mathcal{B}}$ IS THE j-TH COLUMN OF THE PRODUCT $P_{\mathcal{C}}^{\mathcal{E}} P_{\mathcal{E}}^{\mathcal{B}} = \left(P_{\mathcal{E}}^{\mathcal{C}}\right)^{-1} P_{\mathcal{E}}^{\mathcal{B}}$

(b) $$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}_{\mathcal{B}} = [\mathcal{B}]^{-1} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$$

(c) $$[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{\mathcal{B}} [\vec{x}]_{\mathcal{B}} \underset{\text{TECH.}}{=} \begin{bmatrix} 8/3 \\ -1 \\ 1/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 8 \\ -3 \\ 1 \end{bmatrix}$$

CHECK WITH $[\vec{x}]_{\mathcal{C}} = [\mathcal{C}]^{-1} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ ✓

4) Consider two bases $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ and $\mathcal{C} = \{\boldsymbol{c}_1, \boldsymbol{c}_2\}$ of a real vector space $\mathbb{V}$ such that

$$\boldsymbol{b}_1 = 3\boldsymbol{c}_1 + 2\boldsymbol{c}_2 \text{ and } \boldsymbol{b}_2 = -4\boldsymbol{c}_1 + 5\boldsymbol{c}_2.$$

Suppose that $\boldsymbol{x}$ is a vector in $\mathbb{V}$ such that $\boldsymbol{x} = 3\boldsymbol{b}_1 - \boldsymbol{b}_2$, that is $[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

(a) Find $P_{\mathcal{C}}^{\mathcal{B}}$, the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

(b) Compute $[\boldsymbol{x}]_{\mathcal{C}}$.

(a) $P_{\mathcal{C}}^{\mathcal{B}} = \left[[b_1]_{\mathcal{C}} \; [b_2]_{\mathcal{C}} \right] = \begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix}$

(b) $[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{\mathcal{B}} \cdot [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \end{bmatrix}$

5) Consider $\mathcal{C} = \{\boldsymbol{p}_1 = 2x - x^3,\ \boldsymbol{p}_2 = 3 - x, \boldsymbol{p}_3 = x + x^2, \boldsymbol{p}_4 = 1 - x - x^3\}$ in P_3 , the real vector space of polynomials of degree at most 3.

(a) Use the standard basis $\mathcal{E} = \{1, x, x^2, x^3\}$ to prove that $\mathcal{C}$ a basis for P_3 (write down what the definition of a basis is and which theorem you use to justify your answer).

(b) Compute $[2 + x - x^2 + x^3]_{\mathcal{C}}$.

(a) $$[\mathcal{C}] = \left[[\vec{p}_1]_{\mathcal{E}} \cdots [\vec{p}_4]_{\mathcal{E}} \right] = \begin{bmatrix} 0 & 3 & 0 & 1 \\ 2 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} I_4$$

THEN BY THE UNIQUE REPR. TH. $\mathcal{C}$ IS LIN. IND., BUT $\dim P_3 = 4$, THEN $\mathcal{C}$ IS A MAXIMAL SET OF LIN. IND. VECT, THUS A BASIS.

(b) $$[\mathcal{C}] = P^{\mathcal{C}}_{\mathcal{E}} \quad \text{AND} \quad [\vec{x}]_{\mathcal{C}} = [\mathcal{C}]^{-1} [\vec{x}]_{\mathcal{E}} = [\mathcal{C}]^{-1} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \underset{\text{TECH.}}{=}$$

$$[\vec{x}]_{\mathcal{C}} = \frac{1}{4} \begin{bmatrix} 3 \\ 5 \\ -4 \\ -7 \end{bmatrix}$$

6) Let $L: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear mapping defined by

$$L(\boldsymbol{b}_1) = 2\boldsymbol{b}_2 + \boldsymbol{b}_3, \qquad L(\boldsymbol{b}_2) = \boldsymbol{b}_1 + 3\boldsymbol{b}_2 \text{, and } L(\boldsymbol{b}_3) = \boldsymbol{b}_1 + \boldsymbol{b}_2 - \boldsymbol{b}_3,$$

where $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ is a basis of $\mathbb{R}^3$. Find $[L]_{\mathcal{B}}$, the matrix of the linear operator L relative to $\mathcal{B}$.

$$[L]_{\mathcal{B}} = \left[[L(\vec{b}_1)]_{\mathcal{B}} \ [L(\vec{b}_2)]_{\mathcal{B}} \ [L(\vec{b}_3)]_{\mathcal{B}} \right]$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

7) (20 *points*) Let $L: P_2 \longrightarrow P_2$ be the linear mapping defined by

$$L(1+2x) = 3 + x^2, \qquad L(x+x^2) = 2 - x \text{, and } L(3-x^2) = 1 + x + x^2,$$

where $\mathcal{B} = \{\boldsymbol{p_1} = 1 + 2x, \boldsymbol{p_2} = x + x^2, \boldsymbol{p_3} = 3 - x^2\}$ is a basis of P_2, the real vector space of polynomials of degree at most 2. Let $\mathcal{E} = \{1, x, x^2\}$ be the standard basis of P_2.

(a) Find $P_{\mathcal{E}}^{\mathcal{B}}$, the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{E}$.

(b) Find $[L]_{\mathcal{E}}^{\mathcal{B}}$, the matrix of the linear operator L relative to $\mathcal{B}$ and $\mathcal{E}$.

(c) Find $[L]_{\mathcal{B}}$, the matrix of the linear operator L relative to $\mathcal{B}$.

(d) Find $L(x)$.

(a) $$P_{\mathcal{E}}^{\mathcal{B}} = [\mathcal{B}] = \left[[\vec{p}_1]_{\mathcal{E}} \; [\vec{p}_2]_{\mathcal{E}} \; [\vec{p}_3]_{\mathcal{E}} \right] = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

(b) $$[L]_{\mathcal{E}}^{\mathcal{B}} = \left[[L(\vec{p}_1)]_{\mathcal{E}} \; [L(\vec{p}_2)]_{\mathcal{E}} \; [L(\vec{p}_3)]_{\mathcal{E}} \right] = \begin{bmatrix} 3 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(c) $$[L]_{\mathcal{B}} = P_{\mathcal{B}}^{\mathcal{E}} [L]_{\mathcal{E}}^{\mathcal{B}} = \left(P_{\mathcal{E}}^{\mathcal{B}}\right)^{-1} [L]_{\mathcal{E}}^{\mathcal{B}} \underset{\text{TECH}}{=} \frac{1}{5} \begin{bmatrix} -6 & -5 & -1 \\ 12 & 5 & 7 \\ 7 & 5 & 2 \end{bmatrix}$$

(d) NOTE HERE THE VECTOR IS $\vec{p} = x$:

$$[x]_{\mathcal{B}} = P_{\mathcal{B}}^{\mathcal{E}} [x]_{\mathcal{E}} = [\mathcal{B}]^{-1} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \underset{\text{TECH}}{=} \frac{1}{5} \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

$$[L(x)]_{\mathcal{B}} = [L]_{\mathcal{B}} [x]_{\mathcal{B}} \underset{\text{TECH}}{=} \frac{2}{25} \begin{bmatrix} -6 \\ 12 \\ 7 \end{bmatrix}$$

$$[L(x)]_{\mathcal{E}} = P_{\mathcal{E}}^{\mathcal{B}} [L(x)]_{\mathcal{B}} \underset{\text{TECH}}{=} \frac{2}{5} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

THEREFORE: $$L(x) = \frac{2}{5}(3 + x^2)$$

8) (20 *points*) Let $L: P_3 \longrightarrow \mathbb{R}^3$ be the linear mapping defined by

$$L(1) = (1,0,1), \qquad L(x) = (0,-1,1), \qquad L(x^2) = (2,1,-1), \text{and } L(x^3) = (3,1,0).$$

(a) Find a basis for Ker(L).

(b) Find a basis for Range(L).

(c) Use part (a) and (b) to check the Rank-Nullity Theorem.

(d) Specify why L is or is not 1-to-1.

CONSIDER $[L] = [L]_{\mathcal{E}^3}^{\mathcal{E}_3}$ THE MATRIX ASSOCIATED TO L WITH RESPECT TO THE STANDARD BASES $\mathcal{E}_3 = \{1, x, x^2, x^3\}$ AND $\mathcal{E}^3 = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$, THEN: $[L] = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & -1 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$

(a) $\mathrm{Ker}(L) = \mathrm{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \leq P_3$ IF AND ONLY IF $\mathrm{NULL}([L]) = \mathrm{Span}\{\vec{u}_1, \dots, \vec{u}_k\}$ WITH $\vec{u}_i = [\vec{v}_i]_{\mathcal{E}_3} \in \mathbb{R}^4$

$$\mathrm{RREF}([L]) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \mathrm{NULL}([L]) = \mathrm{Span}\{(-1,0,-1,1)\}$$

$$\begin{matrix} t_1 & t_2 & t_3 & t_4 \end{matrix} \rightarrow \begin{cases} t_1 = -t_4 \\ t_2 = 0 \\ t_3 = -t_4 \end{cases} \qquad = \mathrm{Span}\{(1,0,1,-1)\}$$

THUS $\mathrm{Ker}(L) = \mathrm{Span}\{1 + x^2 - x^3\}$ $\left(\text{NOTE: } L(1) + L(x^2) - L(x^3) = 0\right)$

BASIS $= \{1 + x^2 - x^3\}$

(b) $\mathrm{RANGE}(L) = \mathrm{Span}\{\vec{y}_1, \dots, \vec{y}_m\} \leq \mathbb{R}^3$ IFF $\mathrm{COL}([L]) = \mathrm{Span}\{\vec{w}_1, \dots, \vec{w}_m\}$ WITH $[\vec{y}_i]_{\mathcal{E}^3} = \vec{w}_i$.

USING RREF FROM (a), $\mathrm{COL}([L]) = \mathrm{Span}\{L(1), L(x), L(x^2)\} = \mathbb{R}^3$, BECAUSE THESE ARE 3 LIN. IND. VECTORS IN $\mathbb{R}^3$: BASIS $= \mathcal{E}^3$

(c) dim "DOMAIN" = dim Ker(L) + dim RANGE(L): $4 = 1 + 3$ ✓

(d) L IS NOT 1-TO-1 BECAUSE $\dim \mathrm{Ker}(L) \neq 0$.