Mat 320 – Fall 2013 –Exam2

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I certify that I did not receive third party help in *completing* this test (sign) ____________

Instructions. Technology is allowed on this exam. Each problem is worth 10 points. If you use formulas or properties from our book, make a reference. When using technology describe which commands (or keys typed) you used or print out and attach your worksheet.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1) Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -1 & 6 \end{bmatrix}$. Find $4A + B$.

$$4A + B = \begin{bmatrix} 4 & 12 \\ 8 & 24 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} 4+0 & 12+4 \\ 8-1 & 24+6 \end{bmatrix} = \begin{bmatrix} 4 & 16 \\ 7 & 30 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 12 \\ 8 & 24 \end{bmatrix} = \begin{bmatrix} 4\cdot 1 & 4\cdot 3 \\ 4\cdot 2 & 4\cdot 6 \end{bmatrix}$$

2) Find the transpose of the matrix $\begin{bmatrix} 9 & 8 & 9 & 8 \\ 0 & -7 & 0 & -7 \end{bmatrix} = A$

$$A^T = \begin{bmatrix} Col_1(A) \\ Col_2(A) \\ Col_3(A) \\ Col_4(A) \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 8 & -7 \\ 9 & 0 \\ 8 & -7 \end{bmatrix}$$

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3) Consider $\mathcal{B} = \left\{ \overset{B_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \overset{B_2}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \overset{B_3}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$, $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$. Determine if A or C are in *Span*(B).

VECTORS (MATRICES) INSIDE $Span(\mathcal{B})$ ARE; $t_1 B_1 + t_2 B_2 + t_3 B_3 = \begin{bmatrix} t_1 & t_2 \\ t_2 & t_3 \end{bmatrix}$

$A = \begin{bmatrix} t_1 & t_2 \\ t_2 & t_3 \end{bmatrix} \Rightarrow \begin{cases} t_1 = 1 \\ t_2 = 2 \\ t_2 = 2 \\ t_3 = -1 \end{cases}$ CONSISTENT $\Rightarrow A \in Span(\mathcal{B})$

$C = \begin{bmatrix} t_1 & t_2 \\ t_2 & t_3 \end{bmatrix} \Rightarrow \begin{cases} t_1 = 3 \\ t_2 = 2 \\ t_2 = 1 \\ t_3 = 1 \end{cases}$ NOT CONSISTENT $\Rightarrow$ C IS NOT IN $Span(\mathcal{B})$

4) Let $A = \begin{bmatrix} -1 & 0 & -1 \\ 3 & 4 & -2 \end{bmatrix}$ and f_A be the corresponding matrix mapping. Determine the domain, the codomain, and the expression of f_A.

A IS 2×3 THEN DOMAIN $= \mathbb{R}^3$ and CODOMAIN $= \mathbb{R}^2$; $f_A : \mathbb{R}^3 \to \mathbb{R}^2$

$f_A(x_1, x_2, x_3) = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 - x_3 \\ 3x_1 + 4x_2 - 2x_3 \end{bmatrix} = (-x_1 - x_3 ,\ 3x_1 + 4x_2 - 2x_3)$

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5) Suppose that T is a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$ such that $T(\boldsymbol{e_1}) = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$, $T(\boldsymbol{e_2}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, and $T(\boldsymbol{e_3}) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Find a formula for the image of an arbitrary vector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$.

$$[T] = \left[T(\vec{e}_1) \;\; T(\vec{e}_2) \;\; T(\vec{e}_3)\right] = \begin{bmatrix} 6 & 2 & -2 \\ -3 & 0 & 1 \end{bmatrix}$$

$$T(x_1, x_2, x_3) = [T]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (6x_1 + 2x_2 - 2x_3 \,,\, -3x_1 + x_3)$$

6) Suppose that S and T are linear mappings with matrices $[S] = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ and $[T] = \begin{bmatrix} 1 & 4 \\ 2 & 2 \\ -1 & -3 \end{bmatrix}$.

(a) Determine the domain and the codomain of S, and T.
(b) If it is defined determine the matrix that represents $S \circ T$.
(c) Use the expressions of these functions in order to check your result in (b).

(a) $[S]$ IS 3×3 THEN DOMAIN = CODOMAIN = $\mathbb{R}^3$ AND $S: \mathbb{R}^3 \to \mathbb{R}^3$

$[T]$ IS 3×2 THEN DOMAIN = $\mathbb{R}^2$ AND CODOMAIN = $\mathbb{R}^3$ WITH $T: \mathbb{R}^2 \to \mathbb{R}^3$

(b) $S\circ T: \mathbb{R}^2 \xrightarrow{T} \mathbb{R}^3 \xrightarrow{S} \mathbb{R}^3$ IS DEFINED BY $[S\circ T] = [S][T] =$

$$= \begin{bmatrix} 1-5 & 4-15 \\ 2-4 & 2-12 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -4 & -11 \\ -2 & -10 \\ -1 & -3 \end{bmatrix} \Rightarrow [S\circ T]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (-4x_1 - 11x_2 \,,\, -2x_1 - 10x_2 \,,\, -x_1 - 3x_2)$$

(c) $S(y_1, y_2, y_3) = (y_1 + 5y_3 \,,\, y_2 + 4y_3 \,,\, y_3)$

$T(x_1, x_2) = (x_1 + 4x_2 \,,\, 2x_1 + 2x_2 \,,\, -x_1 - 3x_2)$ $\Rightarrow$

$$\Rightarrow S(T(x_1, x_2)) = \left((x_1 + 4x_2) + 5(-x_1 - 3x_2) \,,\, (2x_1 + 2x_2) + 4(-x_1 - 3x_2) \,,\, -x_1 - 3x_2\right)$$

$$= (-4x_1 - 11x_2 \,,\, -2x_1 - 10x_2 \,,\, -x_1 - 3x_2) \quad \checkmark$$

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7) Let $A = \begin{bmatrix} 1 & -2 & 3 & -3 & -1 \\ -2 & 5 & -5 & 4 & 2 \\ -1 & 3 & -2 & 1 & 1 \end{bmatrix}$ and f_A be the corresponding linear mapping. Determine the domain, the co-domain, and a basis for $Ker(f_A)$, the **kernel** (or **nullspace**) of f_A.

$Ker(f_A) = Null(A) =$ "SOLUTION SPACE OF $A\vec{x} = \vec{0}$"

$$RREF(A) = \begin{bmatrix} 1 & 0 & 5 & -7 & -1 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow S = x_3 \begin{bmatrix} -5 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

BASIS OF $NULL(A) = \{(-5,-1,1,0,0), (7,2,0,1,0), (1,0,0,0,1)\}$

8) Let $B = \begin{bmatrix} 1 & -2 & 2 & -3 \\ 2 & -4 & 9 & -2 \\ -3 & 6 & -6 & 9 \end{bmatrix}$ and f_B be the corresponding matrix mapping. Determine the domain, the codomain, and a basis for Range(f_B), the **range** of f_B (or Col(B), the **columnspace** of B).

B IS $3 \times 4 \Rightarrow f_B : \mathbb{R}^4 \to \mathbb{R}^3$

"PIVOTAL COLUMNS" = "NON-FREE VARIABLES"

$Range(f_B) = Col(B) =$ "Span(INDEPENDENT COLUMNS OF B)"

$$RREF(B) = \begin{bmatrix} 1 & -2 & 0 & -4.6 \\ 0 & 0 & 1 & .8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_2, x_4 - FREE VARIABLE $\Rightarrow$ $Col_1 B$ AND $Col_3 B$ ARE LINEARLY INDEPENDENT.

BASIS OF $Col(B) = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ -6 \end{bmatrix} \right\}$

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9) Let $A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \end{bmatrix}$ and f_A be the corresponding linear mapping.

(a) Determine the domain and the codomain of f_A. $\quad f_A : \mathbb{R}^5 \to \mathbb{R}^3$

(b) Determine a basis for Col(A) and one for Null(A).

(c) Use part (b) to verify the Rank Theorem.

(b) $RREF(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \Rightarrow$ GEN. SOL. OF $A\vec{x} = \vec{0}$: $S = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$

FREE FREE $\Rightarrow$ COL_1, COL_2, COL_4 INDEPENDENT.

$NULL(A) = SPAN\{(-1,-2,1,0,0), (1,-3,0,-4,1)\} \Rightarrow \dim(NULL(A)) = 2$

$COL(A) = SPAN\{(1,2,-3), (1,-1,2), (1,1,-2)\} \Rightarrow rank(A) = \dim(COL(A)) = 3$

(c) RANK THEOREM: $\dim(NULL(A)) + rank(A) = \underset{\dim(\text{DOMAIN})}{n}$. HERE $2 + 3 = 5$ ✓

10) Verify that $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ is invertible and use A^{-1} to solve the system $A\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

$RREF([A \mid I_3]) = RREF\left(\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right]\right) =$

$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 4 & -1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$ IT IS I_3; THEN IS A^{-1}.

$A\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \Rightarrow A^{-1}(A\vec{x}) = A^{-1}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \Rightarrow (A^{-1}A)\vec{x} = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -1 & -1 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \Rightarrow$

$\Rightarrow \vec{x} = \begin{bmatrix} -1-1 \\ 4+1-3 \\ -2+3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$

CHECK: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+2+1 \\ -4+2+1 \\ -4+4+3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ ✓