MAT 421 – Exam2 – Fall 2015

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Instructions. Each exercise is worth 10 points.
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Evaluate (if possible) or prove undefined the limit
$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2+2y^2}$$

ALONG $y=0$: $\lim_{x\to 0} \frac{x^2\cdot 0}{x^2} = 0$

ALONG $x=y$: $\lim_{x\to 0} \frac{x^2+\sin^2 x}{x^2+3x^2} = \lim_{x\to 0} \frac{\sin^2 x}{4} = 0$

$\Big]\to$ SEEMS GOING TO 0

ALL SQUARES $\Rightarrow f(x,y)\geq 0$

$2y^2 \geq 0 \Rightarrow x^2 \leq x^2+2y^2$ $\Big] \Rightarrow 0 \leq \frac{x^2}{x^2+2y^2}\sin^2 y \leq \cancel{\frac{x^2+2y^2}{x^2+2y^2}}\sin^2 y$

THEN $0 \leq f(x,y) \leq \sin^2 y$ AND THE BOUNDING FUNCTIONS GO TO 0 AS $(x,y)\to(0,0)$. BY SQUEEZE TH. $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

2. Find the expressions of f_x and f_y for the function
$f(x,y) = e^{xe^y}$

$$f_x = e^y e^{xe^y} = e^{y+xe^y}$$

$$\ln f = xe^y \Rightarrow \frac{f_y}{f} = xe^y \Rightarrow f_y = e^{xe^y}\cdot xe^y = xe^{y+xe^y}$$

LOGARITHMIC DIFFERENTIATION (NOT REALLY NEEDED)

EXP. RULE $f(x,y) = e^{w(x,y)} \Rightarrow f_y = w_y e^w$

$w(x,y) = xe^y \Rightarrow w_y = xe^y$ $\Big] \to f_y = xe^y e^{xe^y}$

3. Find the linearization $L(x,y)$ of the function $f(x,y)=x^3y^4$ at the point $(1,1)$.

$L(x,y) = f_0 + f_x|_0 (x-x_0) + f_y|_0 (y-y_0)$, $(x_0, y_0) = (1,1)$,

$f_x = 3x^2y^4$, $f_y = 4x^3y^3$ $\Rightarrow f_x|_0 = 3$, $f_y|_0 = 4$, $f_0 = 1$

THEN

$L(x,y) = 1 + 3(x-1) + 4(y-1) = 3x + 4y - 6$

4. Find the differential of $u = z - \sin(xy)$.

$du = u_x\,dx + u_y\,dy + u_z\,dz$

$= -y\cos(xy)\,dx - x\cos(xy)\,dy + dz$

5. Find an equation of the tangent plane to parametric surface $\boldsymbol{r}(u,v) = \langle u^2, v^3, uv\rangle$ at $(1,-1,1)$.

VECT EQ.: $\vec{x} = \vec{p} + s\,\vec{r}_u|_{\vec{p}} + t\,\vec{r}_v|_{\vec{p}}$, WHERE $\vec{p} = \langle 1,-1,1\rangle$,

$\vec{r}_u = \langle 2u, 0, v\rangle$, $\vec{r}_v = \langle 0, 3v^2, u\rangle$, THEN $\vec{r} = \vec{p}$ GIVES

$$\begin{cases} u^2 = 1 \\ v^3 = -1 \Rightarrow v = -1 \\ uv = 1 \Rightarrow u = -1 \end{cases} \quad \checkmark\ (-1)^2 = 1$$

THUS $\vec{x} = \vec{p} + s\langle -2, 0, -1\rangle + t\langle 0, 3, -1\rangle$

OR PARAMETRIC EQUATIONS $\begin{cases} x = 1 - 2s \\ y = -1 + 3t \\ z = 1 - s - t \end{cases}$

OR CARTESIAN EQ., WITH $\langle a, b, c\rangle = \vec{r}_u \times \vec{r}_v = \vec{n}$ DIRECTIONAL VECTOR AT POINT $(1,-1,1)$:

$ax + by + cz = \vec{n}\cdot\vec{p}$.

$3x - 2y - 6z = -1$

6. Find the directional derivative of the function $f(x,y,z) = 3x - yz^2$ at the point $(1,-1,1)$ in the direction of $\langle 2,-3,5\rangle$.

GIVEN VECTOR (DIRECTION): $\vec{a} = \langle 2,-3,5\rangle \Rightarrow$ UNIT VECT. $\vec{u} = \frac{1}{\|\vec{a}\|}\vec{a} =$

$= \frac{1}{\sqrt{38}}\langle 2,-3,5\rangle$; $\nabla f = \langle 3, -z^2, -2yz\rangle$

$$D_{\vec{a}} f\big|_{\vec{x}_0} = \vec{u}\cdot\nabla f\big|_{x_0} = \frac{1}{\sqrt{38}}\langle 2,-3,5\rangle\cdot\langle 3,-1,2\rangle$$

$$= \frac{1}{\sqrt{38}}(6+3+10) = \frac{19}{\sqrt{38}} = \frac{\sqrt{38}}{2} \approx 3.08$$

7. Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.

LOOKING FOR THE MINIMUM OF $\overline{OP}$, WHERE $O=(0,0,0)$ AND $P=(x,y,z)$ IS ON SURFACE, THEREFORE WE NEED TO LOOK FOR THE MINIMUM OF $\|P\|$, WHICH IS THE SAME AS THAT OF $\|P\|^2 = x^2+y^2+z^2$. "PLUG THE SURFACE":

$$f(x,z) = x^2 + 9 + xz + z^2.$$

$f_x = 2x + z = 0$

$f_z = x + 2z = 0$

$\Rightarrow \begin{cases} x=0 \\ z=0 \end{cases}$; $f_{xx} = 2$; $f_{xz} = 1$; $f_{zz} = 2$

THEN $D = f_{xx}f_{zz} - f_{xz}^2$

$= 4 - 1 = 3 > 0$

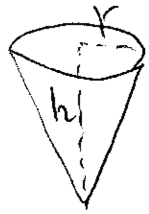
WE HAVE AN EXTREMUM: $f_{xx}(0,0) = 0$ UNDETERMINED, BUT WE CAN CHECK ALONG ONE DIRECTION, SAY $z = x$: $g(x) = f(x,x) =$

$= x^2 + 9 + x^2 + x^2 = 3x^2 + 9 \Rightarrow g''(x) = \frac{d}{dx}[6x] = 6 > 0 \Rightarrow$

$\Rightarrow g$ HAS A MINIMUM $\Rightarrow f(x,z)$ HAS A MINIMUM AT $x=0$, $z=0$.

THIS POINT CORRESPONDS TO THE POINTS ON SURFACE WITH $y^2 = 9$, THE TWO POINTS $(0,3,0)$ AND $(0,-3,0)$ HAVE MIN. DIST. FROM ORIGIN.

8. The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in and the height is 140 in ?



$$V = \frac{1}{3}\pi r^2 h \Rightarrow V' = V_r r' + V_h h'$$
$$V_r = \frac{1}{3}\pi h \cdot 2r \ ; \ V_h = \frac{1}{3}\pi r^2 \ \Rightarrow$$
$$\Rightarrow V' = \frac{1}{3}\pi h \cdot 2r \cdot r' + \frac{1}{3}\pi r^2 \cdot h' = \frac{1}{3}\pi r\left(2hr' + r^2 h'\right)$$

AT THE GIVEN CONDITIONS: $V' = \frac{1}{3}\pi(120)\left(2\cdot 140\cdot 1.8 + (120)(-2.5)\right)$

$= 8160\pi \approx 25635\ in^3/s$, INCREASING VOLUME.

9. Compute the extrema of $f(x,y,z) = 3x - y - 3z$ subject to the constraints $\overbrace{x + y - z}^{g(x,y,z)} = 0$ and $x^2 + 2z^2 = 1$.

$$\nabla f = \lambda \nabla g + \mu \nabla h \quad \begin{cases} 3 = \lambda + 2x\mu \\ -1 = \lambda + 0 \Rightarrow \lambda = -1 \\ -3 = -\lambda + 4z\mu \\ x + y - z = 0 \quad (g) \\ x^2 + 2z^2 = 1 \quad (h) \end{cases}$$

$2x\mu = 4 \Rightarrow x\mu = 2$

$4z\mu = -4 \Rightarrow z\mu = -1$

$z\mu = -1 \Rightarrow z \neq 0$ AND $\mu \neq 0$

IF $\mu \neq 0$:

$$\begin{cases} \lambda = -1 \\ x\mu = 2 \Rightarrow x = 2/\mu \\ z\mu = -1 \Rightarrow z = -1/\mu \\ x^2 + 2z^2 = 1 \Rightarrow \frac{4}{\mu^2} + \frac{2}{\mu^2} = 1 \Rightarrow \mu^2 = 6 \Rightarrow \mu = \pm\sqrt{6} \\ x + y - z = 0 \Rightarrow y = z - x \Rightarrow y = -\sqrt{6}/2 \end{cases}$$

$\mu = \sqrt{6}$	$\mu = -\sqrt{6}$
$x = 2/\sqrt{6} = \sqrt{6}/3$	$x = -\sqrt{6}/3$
$z = -\sqrt{6}/6$	$z = \sqrt{6}/6$
$y = -\sqrt{6}/2$	$y = \sqrt{6}/2$

A) $f\left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right) = 2\sqrt{6}$ $\Rightarrow$ A) MAX.

B) $f\left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right) = -2\sqrt{6}$ $\Rightarrow$ B) MIN.

10. **Doppler effect.** If a sound with frequency f_s is produced by a source travelling along a line with speed v_s and an observer is travelling with speed v_0 along the same line from the opposite direction toward the source, then the frequency of the sound heard by the observer is

$$f_0 = \left(\frac{c + v_0}{c - v_s}\right) f_s$$

where c is the speed of sound, about 332 m/s. Suppose that, at a particular moment, you are in a train travelling at 34 m/s and accelerating at 1.2 $\mathrm{m/s^2}$. A train is approaching you from the opposite direction on the other track at 40 m/s, accelerating at 1.4 $\mathrm{m/s^2}$, and sounds its whistle, which has a frequency of 460 Hz. At that instant, what is the perceived frequency that you hear and how fast is it changing?

AT GIVEN INSTANT: $V_0 = 34$, $V_0' = 1.2$, $V_s = 40$, $V_s' = 1.4$

$f_s = 460$. CONSTANT $c = 332$

$$f_0 = \left(\frac{332 + 34}{332 - 40}\right) 460 \approx 577 \text{ Hz}$$

IT IS ASKED FOR f_0' AT THE GIVEN INSTANT.

$$\frac{d}{dt}[f_0] = V_0' \cdot \partial_{V_0} f_0 + V_s' \cdot \partial_{V_s} f_0 + f_s' \cdot \partial_{f_s} f_0$$

$$= V_0' \frac{1}{c - V_s} f_s + V_s' \cdot f_s \frac{c + V_0}{(c - V_s)^2} + f_s' \cdot \frac{c + V_0}{c - V_s}$$

NOTE THAT f_s MUST BE ASSUMED CONSTANT: $f_s' = 0$

$$= \left(\frac{V_0'}{c - V_s} + \frac{V_s'(c + V_0)}{(c - V_s)^2}\right) f_s$$

$$= \left(\frac{1.2}{332 - 40} + \frac{1.4(332 + 34)}{(332 - 40)^2}\right) \cdot 460$$

$$\approx 4.65 \text{ Hz/s}$$