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Math 321 - Spring 2013 - Test 4

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Instructions. This is an open book exam and you are free to use technology. Each exercise is worth 10 points. **When using a formula report its number and page reference.**
SHOW YOUR WORK NEATLY, PLEASE (no work, no credit).

1. Consider the recursive sequence defined by

$$a_1 = 2, \quad a_{n+1} = 4 - \frac{1}{a_n}.$$

   (a) Prove that for every natural number n we have $2 \le a_n \le 4$.
   (b) Prove that this sequence is monotonic.
   (c) Refer to a Theorem that ensures the convergence of a_n and compute its limit.

(a) I) BY INDUCTION: PROVE $a_n \geq 2$.

BASE: $a_1 = 2 \geq 2$ ✓ ; STEP OF INDUC.: ASSUME $a_n \geq 2$ AND PROVE THAT $a_{n+1} \geq 2$. WRITE: $a_{n+1} = 4 - \frac{1}{a_n}$

$a_n \geq 2 \Rightarrow \frac{1}{a_n} \leq \frac{1}{2} \Rightarrow -\frac{1}{a_n} \geq -\frac{1}{2} \Rightarrow 4 + \left(-\frac{1}{a_n}\right) \geq 4 + \left(-\frac{1}{2}\right)$ ✓

$\Rightarrow a_{n+1} \geq 4 - \frac{1}{2} = 3.5 \geq 2$

II) BY PART I): $a_n \geq 2 > 0 \Rightarrow a_n > 0 \Rightarrow -\frac{1}{a_n} < 0 \Rightarrow \underbrace{4 - \frac{1}{a_n}}_{a_{n+1}} < 4$ ✓

(b) BY INDUCTION: BASE: $a_1 = 2 < a_2 = \frac{7}{2}$ ✓

STEP: ASSUME $a_n < a_{n+1}$ AND PROVE THAT $a_{n+1} < a_{n+2}$:

$a_{n+2} = 4 - \frac{1}{a_{n+1}} \overset{?}{>} a_{n+1} = 4 - \frac{1}{a_n}$. ✓

$a_{n+1} > a_n \Rightarrow \frac{1}{a_{n+1}} < \frac{1}{a_n} \Rightarrow -\frac{1}{a_{n+1}} > -\frac{1}{a_n} \Rightarrow 4 - \frac{1}{a_{n+1}} > 4 - \frac{1}{a_n}$ ✓

(c) BY MONOTONIC BOUNDED SEQUENCES THEOREM, $\{a_n\}$ IS CONVERGENT. SAY $\lim_{n \to \infty} a_n = L$, THEN BY RECURSION $L = 4 - \frac{1}{L} \Rightarrow L^2 = 4L - 1 \Rightarrow$
$\Rightarrow L^2 - 4L + 1 = 0 \Rightarrow L = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$ $\begin{cases} 2-\sqrt{3} < 2 \\ 2+\sqrt{3} \end{cases}$
BECAUSE OF (a) AND (b) $2 \leq L \leq 4$, THEN $L = 2 + \sqrt{3}$.

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2. Use the Integral Test to determine the convergence of the series $\sum_{n=1}^{\infty} \frac{3n^2}{(n^3+1)^2}$.

$$I)\ f(x) = \frac{3x^2}{(x^3+1)^2} \Rightarrow \int_1^{\infty} \frac{3x^2}{(x^3+1)^2}\,dx = \lim_{t\to\infty} \int_2^t u' \cdot u^{-2}\,dx =$$

$u = x^3+1 \to u' = 3x^2$.

NOTE: $x=1 \to u=2$; $x\to\infty$ THEN $u\to\infty$

$$= \lim_{t\to\infty} \left[\frac{u^{-2+1}}{-2+1}\right]_2^t = \lim_{t\to\infty} \left(-\frac{1}{t} - \left(-\frac{1}{2}\right)\right) = 0 + \frac{1}{2} = \frac{1}{2} < \infty$$

II) $f(x)$ IS POSITIVE AND DECREASING FOR $x > 1$

THEN $\to$ OUR SERIES IS CONVERGENT

3. Find the interval of convergence (with endpoints check) of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2n^2}$.

RATIO TEST: $a_n = \frac{(-1)^n x^n}{2n^2} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} x^{n+1}}{2(n+1)^2} \cdot \frac{2n^2}{(-1)^n x^n}\right| =$

$$= \frac{n^2}{(n+1)^2}|x| \xrightarrow[n\to\infty]{} 1 \cdot |x| = |x| < 1 \Rightarrow -1 < x < 1$$

END POINTS: 1) $x = -1 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (-1)^n}{2n^2} = \frac{1}{2}\sum_{n=1} \frac{1}{n^2}$ 2-SERIES CONVERGENT.

2) $x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2}$ THE ABSOLUTE SERIES IS $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{2n^2}\right| = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$

IS CONV. $\Rightarrow$ SERIES ABS. CONV. $\Rightarrow$ SERIES CONV.

INTERVAL OF CONVERGENCE: $-1 \le x \le 1$

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4. Determine the convergence and/or the absolute convergence of the following series (do not find the sum, if it exists). Each part is worth 10 points.

(a) $\sum_{n=2}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$ $\Rightarrow$ ABSOLUTE SERIES $\sum_{n=2}^{\infty}\left|(-1)^{n-1}\frac{n}{n^2+1}\right| = \sum_{n=2}^{\infty}\frac{n}{n^2+1}$

INTEGRAL TEST: $f(x)=\frac{x}{x^2+1}$, CONTINUOUS, POSITIVE, AND DECREASING FOR $x>1$

$$\int_1^{\infty}\frac{x}{x^2+1}\,dx = \int_2^{\infty}\frac{\cancel{x}}{u}\cdot\frac{du}{2\cancel{x}} = \frac{1}{2}\int_2^{\infty}\frac{1}{u}\,du = \lim_{t\to\infty}\left[\ln u\right]_2^{t} =$$

$u = x^2+1 \Rightarrow u' = 2x$

$$= \lim_{t\to\infty}(\ln t-\ln 2) = \infty \Rightarrow \boxed{\text{SERIES IS NOT ABS. CONV.}}$$

ALT. SER. CONV. TEST: $b_n = \frac{n}{n^2+1} = f(n)$ IS DECREASING AND

$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{n^2+1} = 0$ $\Rightarrow$ SERIES IS CONVERGENT

DEG. NUM < DEG DEN

(b) $\sum_{n=0}^{\infty}\frac{2^n+1}{3^n} = \sum_{n=0}^{\infty}\left(\frac{2^n}{3^n}+\frac{1}{3^n}\right) = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$ IS

SUM OF TWO CONVERGENT GEOMETRIC SERIES (BECAUSE $\left|\frac{2}{3}\right|<1$ AND $\left|\frac{1}{3}\right|<1$)

THEREFORE IS CONVERGENT. MOREOVER, IT HAS POSITIVE TERMS

AND IT IS ALSO ABSOLUTELY CONVERGENT.

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5. Use the MacLaurin series for $\ln(x+1)$ to determine the MacLaurin series for the function $h(x) = \dfrac{x}{x^2+1}$.

$$\ln(x+1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$\int h(x)\,dx = \frac{1}{2}\ln(x^2+1) = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x^2)^n}{n} = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n} \Rightarrow$$

$$\Rightarrow h(x) = \frac{d}{dx}\left[\int h(x)\,dx\right] = \frac{d}{dx}\left[\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}\right] =$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{d}{dx}\left[x^{2n}\right] = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(2n)\,x^{2n-1}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}x^{2n-1} = x - x^3 + x^5 - x^7 + x^9 + \cdots$$

6. Find the Taylor series centered at π for $f(x) = \cos x$. First compute the first four Taylor polynomials.

$f^{(0)}(x) = f(x) = \cos x$; $f^{(1)}(x) = f'(x) = -\sin x$; $f^{(2)}(x) = -\cos x$;

$f^{(3)}(x) = \sin x$; $f^{(4)}(x) = \cos x$ (THEN THE "CYCLE" STARTS OVER)

$f^{(i)}(x) = f^{(i+4)}(x)$

i	0	1	2	3	4	---
$f^{(i)}(\pi)$	-1	0	1	0	-1	---

$$T_{f,0,\pi} = -1 = T_{f,1,\pi}\ ;\quad T_{f,2,\pi} = -1 + \frac{1}{2}(x-\pi)^2 = T_{f,3,\pi}$$

$$T_{f,4,\pi} = -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4$$

$$T_{f,\pi} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n)!}(x-\pi)^{2n}$$

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